UNIT - I
MATRICES

CHARACTERISTIC EQUATION:

The equation $|A-\lambda I|=0$ is said to be the characteristic equation of the transformation or the characteristic equation of the matrix A.

Note:

1. characteristic equation for 2×2 matrix is $\lambda^2 - S_1\lambda + S_2 = 0$, where S_1 = sum of the main diagonal elements, $S_2 = |A|$.

2. characteristic equation for 3×3 matrix is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$,

where S_1 = sum of the main diagonal.

S_2 = sum of the minors of the main diagonal. elements

$S_3 = |A|$.

Pbm: 1

Find the characteristic eqn of $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

Soln:

So let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$.

S_1 = sum of the main diagonal $= 1+2 = 3$.

$S_2 = |A| = 2 - 0 = 2$.

$\therefore$ The characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0.$$

2. Find the Characteristic equation of $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Soln:

The characteristic eqn of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

S_1 = Sum of the main diagonal

$= 2 + 1 - 4 = -1$.

S_2 = Sum of the minors of the main diaggonal elements.

$$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$$

$= -4 - 6 - 8 + 5 + 2 + 9$

$= -18 + 16$

$= -2$.

$S_3 = |A|$

$$= \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix} = 2(-4-6) + 3(-12+15) + 1(6+5)$$

$= 2(-10) + 3(3) + 11$

$= -20 + 9 + 11$

$= 0$.

Scanned with OKEN Scanner

$\therefore$ The characteristic equation is

$$\lambda^3 + \lambda^2 - 2\lambda = 0.$$

Eigen Values and Eigen Vectors of a real matrix

Procedure :

Step : 1 Find the characteristic equation $|A - \lambda I| = 0$.

Step : 2 Solving the characteristic equation, we get Eigen values.

Step : 3

To find the eigenvectors, solve,

$$(A - \lambda I)\, x = 0.$$

1. Find the Eigen values and Eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Step : 1

The characteristic equation of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = Sum of the main diagonal.

$= 3 + 5 + 3 = 11$

S_2 = Sum of the minor of the main diagonal.

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$= 15 - 1 + 9 - 1 + 15 - 1$

$= 36.$

$$S_3 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3(15-1) + 1(-3+1) + 1(1-5).$

$= 3(14) + 1(-2) + 1(-4)$

$= 42 - 2 - 4 = 36.$

$\therefore$ The Char. eqn is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$

Step: 2 To find eigen values:

Put $\lambda = 2$, $2^3 - 11(2^2) + 36(2) - 36 = 8 - 44 + 72 - 36$

$= 0.$

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & & 2 & -18 & 36 \\ \hline & 1 & -9 & \cancel{22}\ 18 & \boxed{0} \end{array}$$

$\therefore \lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 6)(\lambda - 3) = 0$

$\therefore \lambda = 6, 3.$

$\therefore$ The eigen values are $\lambda = 2, 3, 6.$

Step: 3

To find eigen vectors, $(A - \lambda I)X = 0.$

i.e., $\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\therefore \left.\begin{aligned} (3-\lambda)x_1 - x_2 + x_3 &= 0 \\ -x_1 + (5-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{— (A)}.$$

Case: (i)

Put $\lambda = 2$ in (A).

$x_1 - x_2 + x_3 = 0$ —— (1)

$-x_1 + 3x_2 - x_3 = 0$ —— (2)

$x_1 - x_2 + x_3 = 0$ —— (3).

Solve (1) and (2).

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}.$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}.$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\therefore X_1 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

· Case : ii)

Put $\lambda = 3$ in (A).

$0x_1 - x_2 + x_3 = 0$ —— (4)

$-x_1 + 2x_2 - x_3 = 0$ — (5)

$x_1 - x_2 + 0x_3 = 0$ —— (6).

Solve (4) & (5).

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$\therefore x_1 = -1, \ x_2 = -1, \ x_3 = -1.$

$\therefore X_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

$\Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (iii)

Put $\lambda = 6$ in (A).

$-3x_1 - x_2 + x_3 = 0$ — (7)

$-x_1 - x_2 - x_3 = 0.$ —— (8)

$x_1 - x_2 - 3x_3 = 0$ —— (9).

Solving (7) & (8).

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}} \Rightarrow \frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$\Rightarrow x_1 = 2, \ x_2 = -4, \ x_3 = 2. \Rightarrow X_3 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

H.W

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Ans: $-2, 3, 6$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

2. Find the eigen values and eigen vectors of $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Ans: $-2, 4, 6$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Properties of Eigen values and Eigen vectors:

Property: 1

(i) The sum of the eigen values of a matrix is equal to the trace of the matrix.

ii) Product of eigen values is equal to the determinant of the matrix.

Property: 2

A square matrix A and its transpose A^T have the same characteristic values.

Property: 3

The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Property: 4

If λ is an eigen value of a matrix A, then $1/\lambda$ $(\lambda \neq 0)$ is the eigen value of A^{-1}.

(5).

1. If λ is an eigen value of a matrix A, what can you say about the eigen value of A^{-1}. Prove your statement

Proof:

w.k.t if x be the eigen vector corresponding to λ, then $Ax = \lambda x$.

Pre multiplying A^{-1} on both sides, we get,

$$A^{-1}Ax = A^{-1}\lambda x.$$

$$\Rightarrow Ix = \lambda A^{-1}x$$

$$\Rightarrow x = \lambda A^{-1}x$$

$$\Rightarrow \frac{1}{\lambda}x = A^{-1}x$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda}x.$$

Hence $1/\lambda$ is an eigen value of the inverse matrix A^{-1}.

Property : 5

If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigenvalue.

Defn:

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$.

Property : 6

The similar matrices have same eigen values.

Property : 7

The eigen vector x of a matrix A is not unique.

Prop : 8

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen values of an $n \times n$ matrix, then corresponding eigen vectors $X_1, X_2, \ldots, X_n$ form a linearly independent set.

Prop : 9

If two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots.

Prop : 10.

If two eigen vector X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$.

Prop : 11.

Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal.

Prop : 12

If A & B are $n \times n$ matrices and B is a non-singular matrix, then A and $B^{-1}AB$ have same eigen values.

Pbms

1. Find the sum and product of the eigen values of the matrix (a) $A = \begin{pmatrix} 2 & -3 \\ 4 & -2 \end{pmatrix}$, b) $B = \begin{pmatrix} 1 & -4 & 4 \\ 1 & -2 & 4 \\ 2 & -1 & 3 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Soln:

a) Sum of the eigen values = Sum of its diagonal elements

$= 2 - 2 = 0$

Product of eigen values $= \begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} = -4 + 12 = 8$

b) Sum of eigen values $= 1 - 2 + 3 = 0$

Product of eigenvalues $= |B| = \begin{vmatrix} 1 & -4 & 4 \\ 1 & -2 & 4 \\ 2 & -1 & 3 \end{vmatrix}$

$= 1(-6+4) + 4(3-8) + 4(-1+4)$

$= 1(-2) + 4(-5) + 4(3)$

$= -2 - 20 + 12$

$= -10$.

c) Sum of eigen values $= 1 + 2 + 1 = 4$

Product of eigen values $= |C| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(2-1) - 2(-1-1) + 3(-1-2)$

$= 1 + 4 - 9 = -4$

2. The product of two eigenvalues of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigen value.

Soln:

Let the eigen values of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ are $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1 \lambda_2 = 16$.

w.k.t $\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 6(9-1)+2(-6+2)+2(2-6) = 6(8)+2(-4)+2(-4)$

$= 48-8-8$

$16 \cdot \lambda_3 = 32$

$\lambda_3 = 2$

3. Two of the eigen values of $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 and 6. Find the eigen values of A^{-1}.

Soln:

Sum of the eigen values $= 3+5+3 = 11$.

Let k be the third eigen value.

$\therefore 3+6+k = 11$

$k = 11-9 = 2$.

Hence the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

4. The eigen vectors of a 3x3 real symmetric matrix A corresponding to the eigen values 2, 3, 6 are $[1, 0, -1]^T$, $[1, 1, 1]^T$ and $[-1, 2, -1]^T$ respectively, find the matrix A.

Soln:

Given Eigen values are 2, 3, 6.

Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

$\sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$

$\sqrt{1^2+1^2+1^2} = \sqrt{3}$

$\sqrt{(-1)^2+2^2+(-1)^2} = \sqrt{6}$

The normalized matrix is,

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}, \quad N^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

W.k.t $D = N^T A N$.

$\Rightarrow ND = NN^T A N$

$\Rightarrow NDN^T = NN^T A NN^T$

Since N is an orthonormal matrix, $NN^T = N^T N = I$.

$\therefore \; NDN^T = A$.

ie, $A = NDN^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/6 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \\ -\sqrt{6} & 2\sqrt{6} & -\sqrt{6} \end{bmatrix}$

$= \begin{bmatrix} 1+1+1 & 0+1-2 & -1+1+1 \\ 0+1-2 & 0+1+4 & 0+1-2 \\ -1+1+1 & 0+1-2 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

Verification:

Sum of the eigen values = Sum of the main diagonal elements.

$2+3+6 = 3+5+3$, ie, $11 = 11$.

Product of eigen values $= |A|$.

$2 \times 3 \times 6 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(15-1) + 1(-3+1) + 1(1-5)$

$36 = 42 - 2 - 4 = 36$.

Cayley - Hamilton Theorem:

Statement:

Every square matrix satisfies its own characteristic equation.

Uses of cayley Hamilton theorem:

(i) To calculate the positive integral powers of A.

ii) To calculate the inverse of a non-singular square matrix A.

Problems:

1. Verify cayley - Hamilton thm, find A^4 and A^{-1} when

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Soln:

The char. eqn of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left((2-\lambda)^2 - 1\right) + 1\left(-1\cdot(2-\lambda)+1\right) + 2\left(1-(2-\lambda)\right) = 0$$

$$(2-\lambda)\left(\overbrace{4-4\lambda+\lambda^2-1}^{3}\right) + 1\left(\overbrace{-2+\lambda+1}^{-1}\right) + 2\left(\overbrace{1-2}^{-1}+\lambda\right) = 0$$

$$6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 - 1 + \lambda - 2 + 2\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 8\lambda + 3 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By cayley - Hamilton thm,

$$A^3 - 6A^2 + 8A - 3I = 0. \longrightarrow (1).$$

Verification:

$$A^2 = A \times A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$+ 8 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29-42+16-3 & -28+36-8 & 38-54+16 \\ -22+30-8+0 & 23-36+16-3 & -28+36-8+0 \\ 22-30+8-0 & -22+30-8 & 29-42+16-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

To find A^4.

(1) $\Rightarrow A^3 = 6A^2 - 8A + 3I$.

Multiply 'A' on both sides,

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6\left[6A^2 - 8A + 3I\right] - 8A^2 + 3A$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$= 28A^2 - 45A + 18I.$$

Reduction of Quadratic form to canonical form by Orthogonal transformation.

Quadratic form:

A homogeneous poly. of second degree in any number of variables is called quadratic form.

Note:

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} \text{coeff } x^2 & \frac{1}{2}\text{ coeff. } x_1x_2 & \frac{1}{2}\text{ coeff } x_1x_3 \\ \frac{1}{2}\text{ coeff } x_2x_1 & \text{Coeff. } x_2^2 & \frac{1}{2}\text{ coeff } x_2x_3 \\ \frac{1}{2}\text{ coeff } x_3x_1 & \frac{1}{2}\text{ coeff } x_3x_2 & \text{coeff } x_3^2 \end{bmatrix}.$$

Index: s

no. of +ve square terms in the canonical form.

Signature: (2s - r).

Difference of no. of +ve and -ve square terms in the canonical form.

Rank: (r)

no. of square terms in the canonical form.

Nature of the Quadratic form:

* Positive definite ⇒ If all the eigen values are +ve.
* +ve semi definite ⇒ If all the eigen values are +ve and atleast one zero.

Scanned with OKEN Scanner

1) Reduce the following symmetric matrix to a diagonal form. $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Soln:

The char. eqn is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 18$

$$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21-16) + (24-4) + (56-36)$$

$$= 5 + 20 + 20 = 45.$$

$S_3 = |A| = 0$.

$\therefore$ The char. eqn is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda = 0, (\lambda - 15)(\lambda - 3) = 0$$

$\Rightarrow \lambda = 0, 15, 3$. are the Eigen values of A.

To find eigen vectors:

Case (i) $\lambda = 0$.

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0.$$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -6 & & 2 & & 8 & & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\therefore \frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}.$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{30}.$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}.$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii) $\lambda = 3$.

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0.$$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 4 & & -4 & & -6 & & 4 & \\ & \times & & \times & & \times & & \\ -4 & & 0 & & 2 & & -4 & \end{array}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{\underset{16}{24-8}}$$

$$x_1 = +2 \quad x_2 = +1 \quad x_3 = -2$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Case (iii) $\lambda = 15$.

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0.$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$X_3 = \cancel{2,-2,1} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

To diagonalize A.

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad |X_1| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3.$$

$$X_2 = \begin{pmatrix} 2 \\ +1 \\ -2 \end{pmatrix}, \quad |X_2| = \sqrt{9} = 3$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad |X_3| = \sqrt{9} = 3.$$

$$\therefore \frac{X_1}{|X_1|} = \begin{pmatrix} \frac{1}{3} \\ 2/3 \\ 2/3 \end{pmatrix}, \quad \frac{X_2}{|X_2|} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \quad \frac{X_3}{|X_3|} = \begin{pmatrix} \frac{2}{3} \\ -2/3 \\ 1/3 \end{pmatrix}.$$

$\therefore$ The normalized modal matrix,

$$\therefore N = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

Now, $$NN^T = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Now, $D = N^T A N$

$$AN = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 8/3 - 12/3 + 4/3 & 16/3 - 6/3 - 4/3 & 16/3 + 12/3 + 2/3 \\ -6/3 + 14/3 - 8/3 & -12/3 + 7/3 + 8/3 & -12/3 - 14/3 - 4/3 \\ 2/3 - 8/3 + 6/3 & 4/3 - 4/3 - 6/3 & 4/3 + 8/3 + 3/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{pmatrix}.$$

$$N^T A N = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 2/3 + 2/3 - 4/3 & 10/3 - 20/3 + 10/3 \\ 0 & 4/3 + 1/3 + 4/3 & 20/3 - 10/3 + -10/3 \\ 0 & 4/3 - 2/3 - 2/3 & 20/3 + 20/3 + 5/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D\,(0, 3, 15).$$

H.W Diagonalize using orthogonal transform $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.

Ans $D(5, -1, -1)$.

Quadratic forms:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

The quadratic form is $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j \quad \ldots \quad (1)$

Pbm:

Obtain the matrices corresponding to the following quadratic form. (i) $x_1^2+2x_2^2-5x_3^2-x_1x_2+4x_2x_3-3x_1x_3$.

(ii) $ax^2+by^2+cz^2+2hxy+2fyz+2gzx$.

Soln:

(i) $$A = \begin{bmatrix} \text{Coeff } x_1^2 & \tfrac12 \text{coeff of } x_1x_2 & \tfrac12 \text{ coeff } x_1x_3 \\ \tfrac12 \text{coef } x_1x_2 & \text{Coeff } x_2^2 & \tfrac12 \text{ coeff } x_2x_3 \\ \tfrac12 \text{ coeff } x_1x_3 & \tfrac12 \text{ coeff } x_2x_3 & \text{coeff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tfrac12 & -\tfrac32 \\ -\tfrac12 & 2 & 2 \\ -\tfrac32 & 2 & -5 \end{bmatrix}$$

(ii) $ax^2+by^2+cz^2+2hxy+2fyz+2gzx$.

$$A = \begin{bmatrix} \text{Coeff } x^2 & \tfrac12 \text{ Coeff } xy & \tfrac12 \text{ coeff } xz \\ \tfrac12 \text{ coeff } xy & \cancel{\tfrac12} \text{ coeff } y^2 & \tfrac12 \text{ Coeff } yz \\ \tfrac12 \text{ coeff } xz & \tfrac12 \text{ Coeff } yz & \cancel{\tfrac12} \text{ coeff } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

2) Write down the quadratic form corresponding to the following matrix.

(i) $\begin{pmatrix} 2 & -3 & 1 \\ -3 & 2 & 4 \\ 1 & 4 & -5 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

Soln:

(i) $$\begin{pmatrix} 2 & -3 & 1 \\ -3 & 2 & 4 \\ 1 & 4 & -5 \end{pmatrix} = 2x_1^2+2x_2^2-5x_3^2-6x_1x_2+2x_1x_3+8x_2x_3$$

(ii) $$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = x_1^2+x_2^2+x_3^2-2x_1x_2+2x_1x_3+2x_2x_3.$$

Definitions:

Rank: (r)

The no. of non-zero eigen values of A

Index: The no. of +ve terms in the canonical form is called the index (P) of the quadratic form.

Signature: (s): The excess of the no. of +ve terms over the no. of negative terms in the canonical form.

ie, $p-(r-p) = 2p-r$.

Nature of quadratic form determined by principal minors:

Let $D_1 = a_{11}$ $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\ldots\ D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

1. $D_1, D_2, \ldots, D_n \geq 0 \Rightarrow$ positive definite
2. $D_i > 0$ and atleast one $D_i = 0 \Rightarrow$ positive semi definite
3. $D_1 < 0,\ D_2 > 0,\ D_3 < 0,\ D_4 > 0 \ldots \Rightarrow$ negative definite
4. $D_1 < 0,\ D_2 > 0,\ D_3 < 0,\ D_4 > 0$ and atleast one $D_i = 0 \Rightarrow$ negative semi definite.
5. In all other cases, the quadratic form is indefinite

1. Find the nature of the following quadratic forms.

(i) $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

Soln:

$$A = \begin{bmatrix} \text{coeff } x_1^2 & \tfrac{1}{2}\text{coeff } x_1x_2 & \tfrac{1}{2}\text{coeff } x_1x_3 \\ \tfrac{1}{2}\text{coeff } x_2x_1 & \text{coeff } x_2^2 & \tfrac{1}{2}\text{coeff } x_2x_3 \\ \tfrac{1}{2}\text{coeff } x_3x_1 & \tfrac{1}{2}\text{coeff } x_3x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$D_1 = 1 > 0$

$D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2-1 = 1 > 0$

$D_3 = |A| = 1(2-1) + 1(-1) = 1-1 = 0$

$\therefore D_1, D_2 > 0, D_3 = 0$

$\therefore$ The quadratic function is +ve semi definite.

3) Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by orthogonal reduction. Also find the rank, index, signature and nature of the quadratic form.

Soln:

The matrix of the given quadratic form is,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

To find eigen values:

$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$

$S_1 = 6+3+3 = 12.$

$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}.$

$= 36$

$S_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32.$

$\therefore$ The char. eqn is $\lambda^3 - 12\lambda + 36\lambda - 32 = 0.$

$\lambda = 2, \; \lambda^2 - 10\lambda + 16 = 0$

$(\lambda-2)(\lambda-8) = 0.$

$\lambda = 2, \lambda = 8.$

$\therefore \lambda = 2, 2, 8$ are the eigen values.

Also the eigen vectors are, $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$

Now, $|X_1| = \sqrt{4+1+1} = \sqrt{6}$

$|X_2| = \sqrt{1+4+0} = \sqrt{5}$

$|X_3| = \sqrt{4+1+25} = \sqrt{30}$.

Now, $N = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix}$ and $N^T = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{pmatrix}$

Clearly $NN^T = N^TN = I$.

$\therefore$ N is orthogonal.

Now, $AN = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0/\sqrt{5} & 5/\sqrt{30} \end{pmatrix}$

$= \begin{pmatrix} 12/\sqrt{6} + 2/\sqrt{6} + 2/\sqrt{6} & \frac{6}{\sqrt{5}} - \frac{4}{\sqrt{5}} + 0 & \frac{-12}{\sqrt{30}} - 2/\sqrt{30} + 10/\sqrt{30} \\ -4/\sqrt{6} - 3/\sqrt{6} - 1/\sqrt{6} & -2/\sqrt{5} + 6/\sqrt{5} - 0 & 4/\sqrt{30} + 3/\sqrt{30} - 5/\sqrt{30} \\ 4/\sqrt{6} + 1/\sqrt{6} + 3/\sqrt{6} & 2/\sqrt{5} - 2/\sqrt{5} + 0 & -4/\sqrt{30} - 1/\sqrt{30} + 15/\sqrt{30} \end{pmatrix}$

$= \begin{pmatrix} 16/\sqrt{6} & 2/\sqrt{5} & -4/\sqrt{30} \\ -8/\sqrt{6} & 4/\sqrt{5} & 2/\sqrt{30} \\ 8/\sqrt{6} & 0 & 10/\sqrt{30} \end{pmatrix}$

$\therefore D = N^T(AN) = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 16/\sqrt{6} & 2/\sqrt{5} & -4/\sqrt{30} \\ -8/\sqrt{6} & 4/\sqrt{5} & 2/\sqrt{30} \\ 8/\sqrt{6} & 0 & 10/\sqrt{30} \end{pmatrix}$

$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

The canonical form is, $8y_1^2 + 2y_2^2 + 2y_3^2$.

Rank = no. of non zero eigen values.

$\therefore \boxed{r = 3}$

Index = no. of +ve eigen values, $\boxed{P = 3}$.

Signature is $s = 2p - r = 2(3) - 3 = 3$

Nature: since the eigen values are positive, the nature is positive definite.

Cayley Hamilton theorem

Statement:

Every square matrix satisfies its own characteristic equation

i.e, $A^n - C_1 A^{n-1} + \dots + (-1)^{n-1} C_{n-1} + (-1)^n C_n I = 0$.

Uses of cayley Hamilton Theorem:

* To find inverse of a matrix A.
* To find higher power of a matrix A.

Problems:

Verify cayley Hamilton theorem for $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ and hence find A^{-1}.

Soln:

To find the char. eqn:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$S_1 = 1+1+1 = 3$

$S_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1-1+1-3+1 = -1$

$S_3 = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1-1) + 3(-2-1) = 3(-3) = -9$

$\therefore$ The char. eqn is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

<u>To prove</u> $A^3 - 3A^2 - A + 9I = 0$. ——— (1).

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}.$$

$$A^3 = A^2 \times A = \begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4-6+6 & 0-3-6 & 12+3+6 \\ 3+4+4 & 0+2-4 & 9-2+4 \\ 0-4+5 & 0-2-5 & 0+2+5 \end{pmatrix} = \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix}$$

$$A^3 - 3A^2 - A + 9I = \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} - 3\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} - 3\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -3 \\ -2 & -1 & +1 \\ -1 & +1 & -1 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ The cayley Hamilton theorem is verified.

<u>To find A^{-1}</u>

Premultiply A^{-1} on both sides in (1).

$\Rightarrow A^{-1}A^3 - 3A^{-1}A^2 - A^{-1}A + 9A^{-1}I = 0.$

$\Rightarrow A^2 - 3A - I + 9A^{-1} = 0.$

$\Rightarrow 9A^{-1} = -(A^2 - 3A - I)$

$\Rightarrow A^{-1} = \frac{-1}{9}(A^2 - 3A + I)$

$$= \frac{-1}{9}\left[\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{-1}{9}\begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}.$$

Pbm:

$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$, find Adj A by using cayley-Hamilton theorem.

Soln:

The characteristic eqn is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

where, $s_1 = 1+1+1 = 3$

$$s_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1 - 1 + 1 + 3 + 1$$

$$= 5$$

$$s_3 = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix} = 1(1-1) - 2(3) - 1(-3) = -6+3 = -3$$

$\therefore$ The char. eqn is $\lambda^3 - 3\lambda^2 + 5\lambda + 3 = 0$

$\therefore$ By cayley Hamilton theorem, $A^3 - 3A^2 + 5A + 3I = 0$.

Pre-multiplying by A^{-1}, we get,

$$A^2 - 3A + 5I + 3A^{-1} = 0.$$

$$\Rightarrow 3A^{-1} = -(A^2 - 3A + 5I).$$

$$\Rightarrow A^{-1} = \frac{-1}{3}(A^2 - 3A + 5I.)$$

Now, $A^2 = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$.

$$= \begin{pmatrix} -2 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{-1}{3}\left[\begin{pmatrix} -2 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 9 & -3 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}\right].$$

$$\therefore A^{-1} = \frac{-1}{3}\begin{pmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{pmatrix}.$$

w.k.t. $A^{-1} = \frac{1}{|A|} \text{adj} A$.

$\Rightarrow \text{adj} A = |A| A^{-1}$

$= -3A^{-1}$

$$= \begin{pmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{pmatrix}.$$

<u>Pbm:</u>

Using Cayley Hamilton theorem find A^{-1} and A^4 for $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$.

<u>Soln:</u>

The char. eqn is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2+2+2 = 6$

$S_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= 4 + 4 - 1 + 4 = 11$

$S_3 : |A| = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 2(4) - 1(2) = 8 - 2 = 6$

$\therefore$ The char. eqn is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

By cayley-Hamilton theorem, $A^3 - 6A^2 + 11A - 6I = 0$.

$\therefore A^2 = A \times A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{pmatrix}$

$A^3 = A^2 \times A = \begin{pmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{pmatrix}$.

To find A^{-1}, premultiply throughout (1) by A^{-1}.

$6A^{-1} = A^2 - 6A + 11I$.

$A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$

$= \frac{1}{6}\left[\begin{pmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{pmatrix} - 6\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} + 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$

$= \frac{1}{6}\left[\begin{pmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} -12 & 0 & 6 \\ 0 & -12 & 0 \\ 6 & 0 & -12 \end{pmatrix} + \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}\right]$

$= \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$.

To find A^4

Multiplying (1) by A.

ie, $A^4 - 6A^3 + 11A^2 - 6A = 0$

$\Rightarrow A^4 = 6A^3 - 11A^2 + 6A = 0$

$\Rightarrow A^4 = 6\begin{pmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{pmatrix} - 11\begin{pmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{pmatrix} + 6\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{pmatrix}$.

Pbm:

Using Cayley Hamilton theorem find $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-4A+I=0$ if the matrix $A=\begin{pmatrix}2&1&1\\0&1&0\\1&1&2\end{pmatrix}$.

Soln:

The char. eqn is $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

$S_1 = 2+1+2=5$

$S_2 = \begin{vmatrix}1&0\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\0&1\end{vmatrix}$

$= 2+3+2 = 7$

$S_3 = \begin{vmatrix}2&1&1\\0&1&0\\1&1&2\end{vmatrix} = 2(2)-1(0)+1(0-1)=3.$

$\therefore$ The char. eqn is $\lambda^3-5\lambda^2+7\lambda-3=0$.

By Cayley Hamilton thm, $A^3-5A^2+7A-3I=0$

Now, $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-4A+I$

$= A^5\left[A^3-5A^2+7A-3I\right]+A^4-5A^3+8A^2-4A+I.$

$= 0+A\left[A^3-5A^2+8A-4I\right)+I.$

$= A\left(A^3-5A^2+7A+A-3I-I\right)+I$

$= A\left(A^3-5A^2+7A-3I\right)+A(A-I)+I$

$= 0+A^2-A+I$

$= A^2-A+I$ —— (1).

Now, $A^2 = \begin{pmatrix}2&1&1\\0&1&0\\1&1&2\end{pmatrix}\begin{pmatrix}2&1&1\\0&1&0\\1&1&2\end{pmatrix} = \begin{pmatrix}4+0+1&2+1+1&2+0+2\\0+0+0&0+1+0&0+0+0\\2+0+2&1+1+2&1+0+4\end{pmatrix}$

$= \begin{pmatrix}5&4&4\\0&1&0\\4&4&5\end{pmatrix}$

$$\therefore A^2 - A + I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 & 3 \\ 0 & 1 & 0 \\ 3 & 3 & 4 \end{pmatrix}$$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 4A + I = A^2 - A + I$$

$$= \begin{pmatrix} 4 & 3 & 3 \\ 0 & 1 & 0 \\ 3 & 3 & 4 \end{pmatrix}.$$

Pbm:

Find the Eigen values and Eigen vectors of the symmetric matrix $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

Soln:

Step 1: To find the Eigen values

The char. eqn is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 4 + 4 + 4 = 12$$

$$S_2 = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15 + 15 + 15 = 45$$

$$S_3 = |A| = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 4(16-1) - 1(4-1) + 1(1-4)$$

$$= 4(15) - 1(3) + 1(-3)$$

$$= 60 - 3 - 3 = 54.$$

$\therefore$ The char. eqn is $\lambda^3 - 12\lambda^2 + 45\lambda - 54 = 0$.

$$\begin{array}{c|cccc} 3 & 1 & -12 & 45 & -54 \\ & 0 & 3 & -27 & 54 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\lambda = 3$, $\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 6)(\lambda - 3) = 0$

$\lambda = 6, 3$.

$\therefore \lambda = 3, 3, 6$ are the eigen values of A.

CHAPTER - II

DIFFERENTIAL CALCULUS

Representation of Functions:

A function is a rule that assigns to each elements 'x' in a set A to each element 'x' in a set 'A' to exactly one element called f(x) in a set 'B'

* Domain :- Let $f : A \rightarrow B$ then set A is called the domain of the function

* Co-Domain :- Set 'B' is called the codomain of the function

* Range :- The set of all images of all the elements of 'A' under the function 'f' is called the range of 'f' and it is denoted by $f(A)$.

Problems:

1) Find the domain of the function $f(x) = \sqrt{x+2}$.

Soln:

Since the square root of a negative number is not defined, the domain of f must be ~~lined~~. positive.

$\therefore x+2 \geq 0$

$\Rightarrow x \geq -2$.

$\therefore x \in [-2, \infty)$.

$\therefore$ Domain is $[-2, \infty)$

2) Find the domain of the function $f(x) = \sqrt{3-x} - \sqrt{2+x}$

Soln:

Let $3 - x \geq 0$ and $2 + x \geq 0$

$\Rightarrow$ $x \leq 3$ and $x \geq -2$

$\therefore$ Domain is $[-2, 3]$.

3. Find the domain of the function $f(x) = \dfrac{x+4}{x^2 - 9}$.

Soln:

The function is not defined at $x^2 - 9 = 0$

$\Rightarrow x^2 = \pm 3$.

$\therefore$ Domain is $\{x \,/\, x \neq 3, x \neq -3\}$.

$\therefore$ Domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

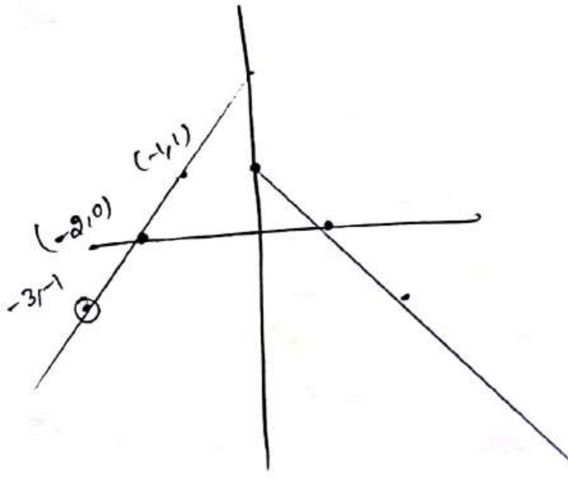
4. Find the domain, range also sketch the graph for the following functions $f(x) = \begin{cases} x+2, & x < 0 \\ 1-x, & x \geq 0. \end{cases}$

Soln:

Given $f(x) = \begin{cases} x+2, & x < 0 \\ 1-x, & x \geq 0. \end{cases}$

Domain x	-1	-2	-3	...
Range $y = x+2$	1	0	-1	...

Domain : x	0	1	2	...
Range : $y = 1-x$	1	0	-1	...



Domain : $(-\infty, \infty)$

Range : $(-\infty, 1)$

H.W Sketch the graph of the fn $f(x) = \begin{cases} 1+x & x < -1 \\ x^2 & -1 \leq x \leq 1 \\ 2-x, & x \geq 1. \end{cases}$

Limit of a function:

Let $f(x)$ be a function of a real variable 'x'. Let 'a' and 'l' be a fixed numbers. If $f(x)$ approaches 'l' as 'x' approaches 'a', then we say 'l' is the limit of $f(x)$ as 'x' tends to 'a' & we write $\lim_{x \to a} f(x) = l$.

Left-hand limit:

If $f(x)$ approaches the value 'l' as 'x' approaches 'a' from the left, then $\lim_{x \to a^-} f(x) = l$

Right-hand limit:

If $f(x)$ approaches the values 'l' as 'x' approaches 'a' from the right, then $\lim_{x \to a^+} f(x) = l$.

Result:

$\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a^-} f(x) = l = \lim_{x \to a^+} f(x)$.

Problems:

1. Evaluate $\lim_{x \to -6} \frac{2x+12}{|x+6|}$

Soln:

w.k.t $\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$.

Given that $\lim_{x \to -6} \frac{2x+12}{|x+6|}$

$$\text{Now, } f(x) = \begin{cases} \frac{2(x+6)}{x+6}, & x+6 \geq 0 \\ \frac{2(x+6)}{-(x+6)}, & x+6 \leq 0. \end{cases}$$

$$= \begin{cases} 2, & x+6 \geq 0 \\ -2, & x+6 < 0 \end{cases}$$

$$= \begin{cases} 2, & x \geq -6 \\ -2 & x < -6. \end{cases}$$

$\therefore \lim_{x \to -6^-} f(x) = -2$ —①

$\lim_{x \to -6^+} f(x) = 2$ —②

$\therefore$ ① $\neq$ ②.

$\therefore$ limit does not exist.

2. Guess the value of the limit (if it exists) from the function $\lim_{x \to 0} \frac{e^{5x}}{x}$ by evaluating the function at the given numbers $x = \pm 0.5$, ± 0.01, ± 0.1, ± 0.001, ± 0.0001 [correct to 6 decimal places]

Soln:

Let $f(x) = \frac{e^{5x} - 1}{x}$.

x	-0.5	-0.1	-0.01	-0.001	-0.0001
$f(x)$	1.835830	3.934693	4.877058	4.987521	4.998750

x	0.5	0.1	0.01	0.001	0.0001
$f(x)$	22.364988	6.487213	5.127110	5.012521	5.001250

As 'x' approaches to '0', the function $f(x) = \frac{e^{5x} - 1}{x}$ approaches to '5'.

$$\therefore \lim_{x \to 0} \frac{e^{5x} - 1}{x} = 5.$$

3) Evaluate $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Soln:

Let $f(x) = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{x^2 \left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{x^2 \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{3}{5}.$$

HORIZONTAL ASYMPTOTE

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Problems:

1. Find the horizontal asymptote of the curve $\frac{x^2-1}{x^2+1}$.

Soln:

Given $f(x) = \frac{x^2-1}{x^2+1}$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2(1 - 1/x^2)}{x^2(1 + 1/x^2)} = 1.$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2(1 - 1/x^2)}{x^2(1 + 1/x^2)} = 1$$

Hence the line $y = 1$ is a horizontal asymptote of the given curve.

2. Find the horizontal and vertical asymptote of the curve $\frac{\sqrt{2x^2+1}}{3x-5}$.

Soln:

Let $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{3x-5} = \lim_{x \to \infty} \frac{x\sqrt{2 + 1/x^2}}{x(3 - 5/x)}$$

$$= \sqrt{2}/3.$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{2x^2-1}}{3x-5} = \lim_{x \to -\infty} \frac{-x\sqrt{2 - 1/x^2}}{x(3 - 5/x)}$$

$$= -\sqrt{2}/3$$

$\therefore$ Both the lines $y = -\sqrt{2}/3$ and $\sqrt{2}/3$ are horizontal asymptotes

The vertical asymptotes occurs when the given function becomes either $-\infty$ or ∞.

For $x = 5/3$, the function becomes ∞.

$$\therefore \lim_{x \to (5/3)^+} f(x) = \lim_{x \to 5/3^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

and
$$\lim_{x \to 5/3^-} f(x) = \lim_{x \to 5/3^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty.$$

SQUEEZE THEOREM:

If $f(x) \le g(x) \le h(x)$, then 'x' is near to 'a' [except the at 'a'] and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Problems:

1. S.T $\lim_{x \to 0} x^2 \sin(1/x) = 0$.

Proof:

Let $f(x) = \lim_{x \to 0} x^2 \sin(1/x)$, $x \ne 0$.

If $x = 0$, $f(x)$ is not defined

If $x \ne 0$, $\frac{1}{x}$ is real.

$\therefore \sin 1/x$ is defined.

$-1 \le \sin 1/x \le 1$

$-x^2 \le x^2 \sin 1/x \le x^2$

$$\lim_{x \to 0} (-x^2) \le \lim_{x \to 0} x^2 \sin 1/x \le \lim_{x \to 0} x^2.$$

Since $\lim_{x \to 0} (-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$

By squeeze theorem,

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

2) S.T $\lim_{x \to 0} \sqrt{x^3+x^2} \sin(\pi/x) = 0$.

Pf:

Let $f(x) = \lim_{x \to 0} \sqrt{x^3+x^2} \sin(\pi/x)$

If $x = 0$, $f(x)$ is not defined.

If $x \ne 0$, π/x is real.

$$-1 \le \sin \pi/x \le 1$$

$$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\,\sin \pi/x \le \sqrt{x^3+x^2}$$

$$\lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\pi/x \le \lim_{x\to 0}\sqrt{x^3+x^2}$$

Since $\lim_{x\to 0} -\sqrt{x^3+x^2} = 0$ & $\lim_{x\to 0}\sqrt{x^3+x^2} = 0$

By squeeze theorem, we have $\lim_{x\to 0}\sqrt{x^3+x^2}\,\sin\pi/x = 0$

H.W: S.T $\lim_{x\to 0} x^2\cos(1/x) = 0$.

Continuous function:

A function $f(x)$ is continuous at $x=a$, if $\lim_{x\to a} f(x) = f(a)$.

Note:

1. If f is continuous at a, then (i) $f(a)$ exist, (ii) $\lim_{x\to a} f(x)$ exist, (iii) $\lim_{x\to a} f(x) = f(a)$.

2. If anyone of above three fail, then $f(x)$ is discontinuous at $x=a$.

Problems:

1. S.T the following functions are discontinuous.

i) $f(x) = \frac{1}{x+2}$, $a=-2$, (ii) $f(x) = \begin{cases} e^x & \text{if } x<0 \\ x^2 & \text{if } x\ge 0 \end{cases}$, $a=0$

(iii) $f(x) = \begin{cases} \cos x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1-x^2 & \text{if } x>0 \end{cases}$, $a=0$.

Soln:

1. $a=-2$.

$$f(-2) = \frac{1}{-2+2} = \frac{1}{0} = \infty.$$

$f(-2)$ does not exist

$\therefore f(x)$ is discontinuous.

ii) Here $a = 0$.

$$f(0) = \begin{cases} e^{0}, & x < 0 \\ 0^{2}, & x \geq 0 \end{cases} = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}.$$

limit does not exist.

$\therefore$ $f(x)$ is discontinuous.

iii) $a = 0$,

$$f(0) = \begin{cases} \cos 0, & x < 0 \\ 0, & x = 0 \\ 1 - 0, & x > 0 \end{cases} = \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

$f(0) = 0$ but $\lim\limits_{x \to 0} f(0) = 1$.

$\therefore$ $f(x)$ is discontinuous.

2. For what value of the constant 'c' is the function 'f' continuous On $(-\infty, \infty)$, $f(x) = \begin{cases} cx^2 + 2x & x < 2 \\ x^3 - cx & x \geq 2. \end{cases}$

Soln:

For every value of 'c', $f(x)$ is continuous on $x < 2$ & $x \geq 2$.

At $x = 2$

$f(x)$ is continuous if $cx^2 + 2x = x^3 - cx$ $(x = 2)$

$$\Rightarrow 4c + 4 = 8 - 2c$$

$$\Rightarrow 6c = 4$$

$$\Rightarrow \boxed{c = 2/3}.$$

$f(x)$ is continuous at all point x, if $c = 2/3$.

3) where are each of the following functions discontinuous?

i) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (ii) $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

iii) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$

Soln:

i) Given $f(x) = \frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{x-2}$

$= x+1$

i.e, $f(x) = x+1$, which is a polynomial.

$\therefore$ $f(x)$ is continuous at all points.

ii) $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Now, $\lim_{x \to 0} f(x)$ doesnot exist

$\therefore$ $f(x)$ is not continuous at $x = 0$.

iii) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2} x+1 = 3.$

But $f(2) = 1$.

$\therefore \lim_{x \to 2} f(x) \neq f(2)$.

$\therefore$ f is not continuous at $x = 2$.

Differentiation:

The derivative of a function $f(x)$ at a number 'a' denoted by $f'(a)$ & is defined by,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ if this limit exist.

Problems:

1. If $f(x) = \sqrt{x}$, find the derivative of $f(x)$.

Soln:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

2. If $f(x) = \sin x$, find the derivative of $f(x)$.

Soln:

Let $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin h/2}{h} \quad \left[\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \cos(x + h/2)$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \times \lim_{h \to 0} \cos(x + h/2)$$

$$= 1 \times \lim_{h \to 0} \cos x \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \cos x.$$

Rules of Differentiation:

Product rule: $\frac{d}{dx}[uv] = vu' + uv'$

Quotient rule: $\frac{d}{dx}[u/v] = \frac{vu' - uv'}{v^2}$

Chain rule: Given $F(x) = f[g(x)]$.

Take $g(x) = u$. Then $F(x) = f(u)$.

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Problems:

1. Differentiate the following functions.

i) $1/x^2$ ii) $\sqrt[3]{x^2}$ iii) $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 7$.

iv) $y = ax^{2n} + bx^n + c$ v) $y = \frac{x^3+4x^2+3}{x^2}$ vi) $y = e^x - x$.

vii) $y = \frac{xe^x - 1}{x}$ viii) $y = a^x$.

Soln:

i) $y = 1/x^2 = x^{-2}$

$$\frac{dy}{dx} = -2x^{-3} = -2/x^3.$$

ii) $y = \sqrt[3]{x^2} = x^{2/3}$

$$\frac{dy}{dx} = 2/3\, x^{2/3-1} = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$$

iii) $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 7$

$$\frac{dy}{dx} = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

iv) $y = ax^{2n} + bx^n + c$

$$\frac{dy}{dx} = 2na\,x^{2n-1} + nbx^{n-1}.$$

v) $y = \frac{x^3+4x^2+3}{x^2} = \frac{x^3}{x^2} + \frac{4x^2}{x^2} + 3/x^2$

$$y = x + 4 + 3x^{-2}$$

$$\frac{dy}{dx} = 1 + 3(-2)x^{-3}$$

$$= 1 - 6/x^3.$$

vi) $y = e^x - x \Rightarrow dy/dx = e^x - 1$

vii) $\frac{dy}{dx} y = \frac{xe^x - 1}{x} = e^x - 1/x.$

$$\frac{dy}{dx} = e^x + 1/x^2$$

viii) $y = a^x = e^{\log a^x} = e^{x\log a}$

$$\frac{dy}{dx} = e^{x\log a} . \log a = a^x . \log a.$$

2) Differentiate the following functions.

i) $f(x) = (x^3+2x)\, e^x$ (ii) $f(x) = (1-e^x)(x+e^x)$

iii) $f(x) = \dfrac{x^2+x-2}{x^3+6}$ iv) $f(x) = \dfrac{e^x}{x}$.

v) $y = (1-x^2)^{10}$ vi) $y = \left(\dfrac{x^2+1}{x^2-1}\right)^3$

Ans:

i) Given $f(x) = (x^3+2x)\, e^x$

we apply product rule.

$$\frac{d}{dx}[uv] = vu' + uv'.$$

Here $u = x^3+2x$ $v = e^x$

$u' = 3x^2+2$ $v' = e^x$

$$\frac{d}{dx}[f(x)] = e^x(3x^2+2) + (x^3+2x)\, e^x.$$

ii) $f(x) = (1-e^x)(x+e^x)$.

$$\frac{d}{dx}[uv] = vu' + uv'$$

$u = 1-e^x$ $v = x+e^x$

$du = -e^x$ $v' = 1+e^x$

$$\frac{d}{dx}(f(x)) = (x+e^x)(-e^x) + (1-e^x)(1+e^x)$$

$$= -xe^x - e^{2x} + 1 - e^{2x}$$

$$= 1 - xe^x - 2e^{2x}$$

iii) $f(x) = \dfrac{x^2+x-2}{x^3+6}$.

We apply quotient rule,

$$\frac{d}{dx}[u/v] = \frac{vu' - uv'}{v^2}$$

$u = x^2+x-2$ $v = x^3+6$

$u' = 2x+1$ $v' = 3x^2$.

$u = x^2 + x - 2 \qquad v = x^3 + 6$

$u' = 2x + 1 \qquad v' = 3x^2$

$$\frac{d}{dx}\left[\frac{x^2+x-2}{x^3+6}\right] = \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4 + 12x + x^3 + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

iv) $f(x) = \frac{e^x}{x}$

$$\frac{d}{dx}\left[u/v\right] = \frac{vu' - uv'}{v^2}$$

$u = e^x \qquad v = x$

$du = e^x \qquad dv = \cancel{dx}\ 1$

$$\frac{d}{dx}\left[\frac{e^x}{x}\right] = \frac{xe^x - e^x}{x^2}.$$

v) $y = (1-x^2)^{10}$

Let $u = 1 - x^2 \Rightarrow y = u^{10}$.

$\frac{dy}{du} = 10u^9$ and $\frac{du}{dx} = -2x$.

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10u^9 \times (-2x)$$

$$= 10(1-x^2)^9(-2x)$$

vi) $y = \left(\frac{x^2+1}{x^2-1}\right)^3$

$\frac{dy}{dx}$ Let $u = \frac{x^2+1}{x^2-1}$.

$\Rightarrow y = u^3$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Now, $\frac{dy}{du} = 3u^2$, $\frac{du}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$

$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$.

$\therefore \frac{dy}{dx} = 3u^2 \cdot \left(\frac{-4x}{(x^2-1)^2}\right)$

$= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \left(\frac{-4x}{(x^2-1)^2}\right)$.

3) Find $\frac{dy}{dx}$ from $x^2+y^2 = a^2$

Soln:

Gn, $x^2+y^2 = a^2$.

$2x + 2y \cdot \frac{dy}{dx} = 0$

$\cancel{2}y \frac{dy}{dx} = -\cancel{2}x$

$\frac{dy}{dx} = -x/y$

4) Find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Soln:

Given $x^{1/2} + y^{1/2} = a^{1/2}$.

Diff w.r.to 'x',

$\frac{1}{2} x^{-1/2} + \frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx} = 0$.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}.$$

5) Find y'' if $9x^2+y^2=9$.

Soln:

Gn, $9x^2+y^2=9$.

Diff w.r.t 'x'

$$18x + 2y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-18x}{2y} = -9x/y.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-9x}{y}\right)$$

$$\frac{d}{dx}(u/v) = \frac{vu'-uv'}{v^2}$$

Here $u=-9x$ $\quad v=y$

$u'=-9$ $\quad v'=y'$.

$$y'' = y(-9) - (-9x)y'$$

$$= -9y + 9xy'$$

$$y'' = 9(xy'-y)$$

H.W

Find y'' if $\sqrt{x}+\sqrt{y} = \sqrt{a}$.

Ans: $\cancel{y}\ \dfrac{\sqrt{x}+\sqrt{y}}{2\sqrt{x}\cdot x}$.

6) Find the tangent line to the equation $x^3+y^3=6xy$ at the point (3,3) and at what point the tangent line horizontal in two first quadrant.

Ans

Given $x^3+y^3=6xy$.

Tangent line eqn $y-y_1 = \left(\frac{d}{dx}\right)_{(x_1,y_1)}(x-x_1)$

$$y-y_1 = y'_{(x_1,y_1)}(x-x_1).$$

Given, $x^3+y^3=6xy$

$$3x^2+3y^2\cdot\frac{dy}{dx} = 6x\cdot\frac{dy}{dx}+6y.$$

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx} = 6y-3x^2$$

$$\frac{dy}{dx}(3y^2-6x) = (6y-3x^2) \Rightarrow \frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$$

At (3,3) $\frac{dy}{dx} = \frac{18-27}{27-18} = \frac{-9}{9} = -1$

$\therefore$ Tangent line equation is,

$y-3 = -1(x-3).$

$y-3 = -x+3$

$\Rightarrow y = -x+6.$

Horizontal tangent,

$$\frac{dy}{dx}=0 \Rightarrow \frac{6y-3x^2}{3y^2-6x}=0.$$

$\Rightarrow 6y-3x^2=0$

$\Rightarrow 6y=3x^2 \Rightarrow x^2=2y \Rightarrow x=\sqrt{2y}.$

$\therefore$ ① $\Rightarrow (\sqrt{2y})^3 + y^3 = 6(\sqrt{2y})\,y$

$2\sqrt{2}\cdot y\sqrt{y} + y^3 = \cancel{12}\ 6y\sqrt{2}\sqrt{y}$.

$2\sqrt{2}\,y^{3/2} + y^3 - 6\sqrt{2}\,y^{3/2} = 0$

$y^{3/2}\left(2\sqrt{2} + y^{3/2} - 6\sqrt{2}\right) = 0$

$y^{3/2}\left(y^{3/2} - 4\sqrt{2}\right) = 0$

$y^{3/2} = 4\sqrt{2}$

$y = (4\sqrt{2})^{2/3}$

$= (32)^{1/3}$

$= 2\sqrt[3]{4}$

$\Rightarrow x = \sqrt{2y} = (2y)^{1/2} = \left(2\cdot 2\cdot \sqrt[3]{4}\right)^{1/2}$

$= 2(4)^{1/6}$

$= 2(2^2)^{1/6}$

$= 2\cdot 2^{1/3}$

$= 2\sqrt[3]{2}$

$\therefore$ The point is $\left(2\sqrt[3]{2},\ 2\sqrt[3]{4}\right)$.

7) Find the eqn of two tangent line to the parabola $y = x^2 - 8x + 9$ at $(3, -6)$.

Soln:

Given $y = x^2 - 8x + 9$ $\quad (x_1, y_1) = (3, -6)$.

Now, $\frac{dy}{dx} = 2x - 8$

$\frac{dy}{dx}(3, -6) = 2(3) - 8 = 6 - 8 = -2$.

Tangent line : $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

$$y + 6 = (-2)(x - 3).$$

$$y + 6 = -2x + 6$$

$$y = -2x + 6 - 6$$

$$\boxed{y = -2x}$$

7) Find an eqn of a tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

Ans:

$$y = x\sqrt{x} = x^{3/2}$$

$$\frac{dy}{dx} = 3/2\, x^{3/2 - 1}$$

$$= 3/2\, x^{1/2}$$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3/2\; 1^{1/2} = 3/2.$$

Tangent line :

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$$y - 1 = 3/2\,(x - 1)$$

$$y - 1 = 3/2\, x - 3/2$$

$$y = 3/2\, x - 3/2 + 1$$

$$y = 3/2\, x - 1/2.$$

Normal line :

$$y - y_1 = \frac{-1}{\left(dy/dx\right)_{(x_1, y_1)}} (x - x_1).$$

$$y - 1 = \frac{-1}{3/2} (x - 1)$$

$$y - 1 = -2/3\,(x - 1).$$

$$y - 1 = -2/3\, x + 2/3$$

$$y = -2/3\, x + 2/3 + 1.$$

$$y = -2/3\, x + 5/3.$$

Scanned with OKEN Scanner

8). Find the eqn of two tangent line to the $y = \frac{e^x}{1+x^2}$ at two point $(1, \frac{1}{2}e)$

Ans:

Given $y = \frac{e^x}{1+x^2}$ $u = e^x$ $v = 1+x^2$

$u' = e^x$ $v' = 2x$.

$$\frac{dy}{dx} = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$$

$$\left(\frac{dy}{dx}\right)_{(1, e/2)} = \frac{(1+1^2)e^1 - e^1(2)}{(1+1^2)^2}$$

$$= \frac{2e^1 - 2e^1}{4} = 0.$$

Tangent line:

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$$y - e/2 = 0(x-1)$$

$$\boxed{y = e/2}$$

Parametric form:

If $x = f(t)$ & $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

Pbms:

1) Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$

soln:

$x = at^2$ $y = 2at$

$\frac{dx}{dt} = 2at$ $\frac{dy}{dt} = 2a$.

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2a}{2at} = 1/t.$$

2) Find $\frac{dy}{dx}$ if $x = a(\cos\theta + \log\tan\theta/2)$, $y = a\sin\theta$.

Soln:

$x = a(\cos\theta + \log\tan\theta/2)$ $\quad\quad$ $y = a\sin\theta$

$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\theta/2}\sec^2\theta/2 \cdot 1/2\right)$; $\quad \frac{dy}{d\theta} = a\cos\theta$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{a\left(-\sin\theta + \frac{\sec^2\theta/2}{2\tan\theta/2}\right)}.$$

Derivative using logarithmic:

1) Find $\frac{dy}{dx}$ when $y = (\tan x)\log x$

Soln:

Given $y = (\tan x)^{\log x}$

Take log on both sides,

$$\log y = \log(\tan x)^{\log x}$$

$$\log y = \log x \log(\tan x)$$

Diff. w.r.t x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = (1/x)\log(\tan x) + \log x \frac{1}{\tan x}\sec^2 x$$

$$\frac{dy}{dx} = y/x \log(\tan x) + y\log x\left(\frac{\sec^2 x}{\tan x}\right).$$

2) Find $\frac{dy}{dx}$ when $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$.

Soln:

$$y = (\tan x)^{\cot x} + (\cot x)^{\tan x} \quad \text{——} (1)$$

Let $u = (\tan x)^{\cot x}$, $\quad v = (\cot x)^{\tan x}$

Taking 'log' on both sides, we get,

$$\log u = \log(\tan x)^{\cot x}$$

$$\log u = \cot x \log(\tan x)$$

$$\frac{1}{u}\frac{du}{dx} = \cot x\frac{1}{\tan x}\sec^2 x + \log(\tan x)(-\operatorname{cosec}^2 x)$$

Scanned with OKEN Scanner

$$\frac{du}{dx} = u\left[\cot^2 x \sec^2 x - \operatorname{cosec}^2 x \log(\tan x)\right]$$

$$\frac{du}{dx} = (\tan x)^{\cot x}\left[\cot^2 x \sec^2 x - \operatorname{cosec}^2 x \log(\tan x)\right]$$

Here $v = (\cot x)^{\tan x}$

$$\log v = \tan x \log(\cot x)$$

$$\frac{1}{v}\frac{dv}{dx} = \tan x \frac{1}{\cot x}\left[-\operatorname{cosec}^2 x\right] + \log(\cot x)\sec^2 x$$

$$\frac{dv}{dx} = v\left[-\tan^2 x \operatorname{cosec}^2 x + \sec^2 x \log(\cot x)\right]$$

$$\frac{dv}{dx} = (\cot x)^{\tan x}\left[-\tan^2 x \operatorname{cosec}^2 x + \sec^2 x \log(\cot x)\right]$$

$$(1) \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

$$= (\tan x)^{\cot x}\left[\cot^2 x \sec^2 x - \operatorname{cosec}^2 x \log(\tan x)\right]$$

$$+ (\cot x)^{\tan x}\left[-\tan^2 x \operatorname{cosec}^2 x + \sec^2 x \log(\cot x)\right].$$

3) Find y' if $y = (\log x)^{\sin x}$.

Soln:

Gn, $y = (\log x)^{\sin x}$

$$\log y = \log(\log x)^{\sin x}$$

$$\log y = \sin x \log(\log x))$$

Diff w.r.to x,

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\cos x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x\log x} + \cos x \log(\log x)\right)$$

$$= (\log x)^{\sin x}\left[\frac{\sin x}{x\log x} + \cos x \log(\log x)\right]$$

4) Find y' if $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdots}}}$

Ans:

Gn, $y = (\sin x)^y$

$\log y = y \log \sin x$.

$$\frac{1}{y}\frac{dy}{dx} = y\left[\frac{1}{\sin x}\cos x\right] + \log(\sin x)\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - \log(\sin x)\frac{dy}{dx} = y\cot x.$$

$$\frac{dy}{dx}\left[\frac{1}{y} - \log(\sin x)\right] = y\cot x.$$

$$\frac{dy}{dx} = \frac{y\cot x}{\frac{1}{y} - \log(\sin x)}$$

Applications: Maxima and minima of functions of One Variable:

Let 'c' be a point in a Domain 'D' of the function 'f'. Then $f(c)$ is the absolute maximum value of 'y' on 'D', if $f(c) \geq f(x)$, for all 'x' in D, absolute minimum value of 'f' on 'D' if $f(c) \leq f(x)$ for all x in 'D'.

Absolute maximum and minimum of f(x):

1. Find the critical number of $f(x)$, ie $f'(x) = 0$.
2. Substitute the critical points at $f(x)$.
3. Then the maximum value of $f(x)$ is absolute maximum and the minimum value of $f(x)$ is absolute minimum.

Problems:

1. Find the absolute maximum and absolute minimum of $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

Soln:

Given $f(x) = x - 2\sin x$.

$\therefore f'(x) = 1 - 2\cos x.$

Critical numbers : $f'(x) = 0$

$\Rightarrow 1 - 2\cos x = 0$

$\cos x = 1/2$

$x = \pi/3, 5\pi/3.$

$\therefore f(\pi/3) = \pi/3 - 2\sin(\pi/3) = -0.68485$

$f(5\pi/3) = \frac{5\pi}{3} - 2\sin(5\pi/3) = 6.968039.$

At end points, $f(0) = 0 - 2\sin 0 = 0$

$f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi = 6.28.$

$\therefore$ The absolute minimum value is $f(\pi/3) = -0.68485$

" " maximum value is $f(5\pi/3) = 6.9680$

2) Find the absolute maximum and absolute minimum value of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

Soln:

Gn, $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$f'(x) = 12x^3 - 12x^2 - 24x.$

$\therefore$ The critical numbers of $f(x)$ are occured at,

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$x = 0, 2, -1.$

The values of $f(x)$ at these critical numbers are,

$f(0) = 0 + 1 = 1.$

$f(2) = 3(16) - 4(8) - 12(4) + 1 = -4.$

$f(-1) = 3(1) - 4(-1) - 12(1) + 1 = -31.$

At end points, $f(-2) = 33$ and $f(3) = 28.$

$\therefore$ The absolute maximum value is $f(-2) = +33$

" minimum value is $f(+2) = -31$

FIRST AND SECOND DERIVATIVE TEST

* If $f'(x) > 0$ in an interval (a, b), then 'f' is increasing
* If $f'(x) < 0$ in an interval (a, b), then 'f' is decreasing

Problems:

1. The profit function of acrosonic is given by $P(x) = -0.02x^2 + 300x - 200000$, dollars, where the function 'p' is decreasing and where it is decreasing.

Soln:

The derivative 'p' of the function p is,

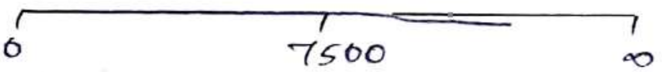
$$p'(x) = -0.04x + 300$$

$$p'(x) = -0.04(x - 7500)$$

To find critical points, $p'(x) = 0$.

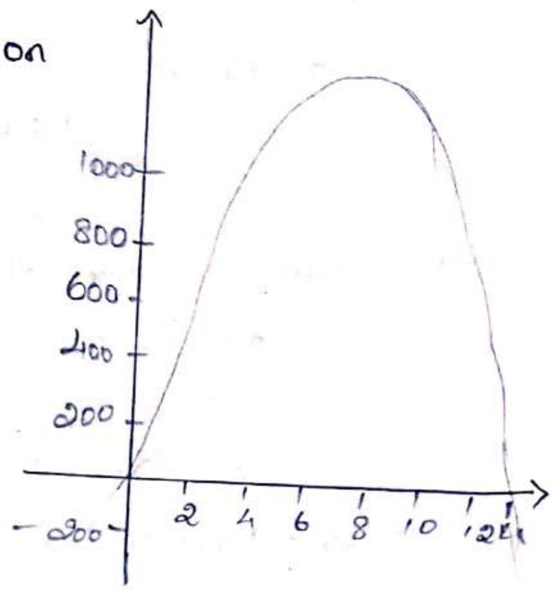
$$-0.04(x - 7500) = 0$$

$$\Rightarrow x = 7500$$



* $p'(x) > 0$, for 'x' in the interval $(0, 7500)$
* $p'(x) < 0$ for 'x' in the interval $(7500, \infty)$

This means that the profit function 'p' is increasing on $(0, 7500)$ and decreasing on $(7500, \infty)$.



2) For the function $f(x) = 2+2x^2-x^4$, find the intervals of increasing (or) decreasing, local maximum, (or) local minimum values and the intervals of concavity, also the inflexion points.

Soln:

Given $f(x) = 2+2x^2-x^4$

$$f'(x) = 4x - 4x^2$$

To find the critical pts,

$$f'(x)=0 \Rightarrow 4x-4x^3=0$$

$$4x(1-x^2)=0$$

$$\Rightarrow x=0,\ x=\pm 1$$

The critical pts are $-1, 0, 1$.

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty,-1)$	$f'(-2) = 4(-2)-4(-2)^3 = 24$ (+ve)	increasing
$(-1,0)$	$f'(-1/2) = 4(-1/2)-4(-1/2)^3 = -3/2$ (−ve)	Decreasing
$(0,1)$	$f'(1/2) = 4(1/2)-4(1/2)^3 = 3/2$ (+ve)	Increasing
$(1,\infty)$	$f'(2) = 4(2)-4(2)^3 = -24$	Decreasing

$f(x)$ has local maximum at $x=-1$.

$$\therefore f(-1) = 2+2(-1)^2-(-1)^4 = 3.$$

$f(x)$ has local minimum at $x=0$

$$\therefore f(0) = 2+2(0)-0 = 2$$

$f(x)$ has local maximum at $x=1$.

$$f(1) = 3$$

Concavity:

$f''(x) = 4 - 12x^2$

$f''(x) = 0 \Rightarrow 12x^2 = 4 \Rightarrow x^2 = 1/3 \Rightarrow x = \pm 1/\sqrt{3} = \pm 0.58$

$-\infty \quad -0.58 \quad 0.58 \quad \infty$

Interval	Sign of $f''(x)$	Behaviour of $f(x)$
$(-\infty, -0.58)$	$f(-1) = 4 - 12(-1)^2 = -8$ (−ve)	Concave downward
$(-0.58, 0.58)$	$f(0) = 4$ (+ve)	concave upward
$(0.58, \infty)$	$f(1) = -8$ (−ve)	concave downward.

$f'(-1) = 0$ and $f''(-1) = -8 < 0$

$\therefore f(x)$ is local maximum at $x = -1$.

$f'(0) = 0$ and $f''(0) = 4 - 12(0) = 4 > 0$.

$\therefore f(x)$ is local mini minimum at $x = 0$

$f'(1) = 0$, $f''(1) = -8 < 0$.

$\therefore f(x)$ is local maximum at $x = 1$.

Inflection points:

* Curve changes from concave downward to concave upward at $x = -0.58$.

$\therefore f(-0.58) = 2 + 2(-0.58) - (0.58)^4$

$= 2.56$

Inflection pts are $(-0.58, 2.56)$

Also curve changes from concave upward to concave downward at $x = 0.58$.

$\therefore f(0.58) = 2 + (0.58) - (0.58)^4 = 2.56$.

$\therefore$ Inflection points are $(0.58, 2.56)$.

CHAPTER - III

FUNCTIONS OF SEVERAL VARIABLES.

PARTIAL DIFFERENTIATION

If $z = f(x,y)$ be a function of two variables x & y and if we keep y as constant and vary x alone, then z is a function of x only. The derivative of z w.r.to x and it is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$ or f_x.

Note: ① $f(x) = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$

$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$

② $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Problems:

① If $u = (x-y)(y-z)(z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln:

Given: $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$

$$= -(y-z)(x-y) + (z-x)(y-z)$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$= (z-x)(x-y) - (z-x)(y-z)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (y-z)(-1)\right]$$

$$= (z-x)(x-y) - (z-x)(y-z)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right]$$

$$= (x-y)(y-z) - (x-y)(z-x).$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

2. If $u = x^y$, then show that (i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$.

Soln:

Given, $u = x^y = e^{\log x^y} = e^{y \log x}$

$$\frac{\partial u}{\partial y} = e^{y\log x} . \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{y\log x}(1/x) + \log x \, e^{y \log x} y(1/x).$$

$$= \frac{x^y}{x} + \log x \frac{x^y}{x} \cdot y$$

$$= x^{y-1} + x^{y-1} y \log x$$

$$= x^{y-1}(1 + y\log x] \quad \text{—(1)}$$

Now, $\frac{\partial u}{\partial x} = e^{y\log x} y(1/x) = e^{y\log x} \cdot y/x$.

$$\frac{\partial^2 u}{\partial y \partial x} = e^{y\log x}(1/x) + y/x \, e^{y\log x} \log x$$

$$= \frac{x^y}{x} + \frac{x^y}{x} y \log x$$

$$= x^{y-1} + x^{y-1} y \log x$$

$$= x^{y-1}(1 + y\log x) \quad \text{—(2)}.$$

From (1) & (2) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$

Diff. partially w.r.to 'x' on both sides,

$$U_{xxy} = u_{xyx}.$$

3. If $f(x,y) = \log\sqrt{x^2+y^2}$, show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Soln:

Gin, $f = \log\sqrt{x^2+y^2}$

$$= \frac{1}{2}\log(x^2+y^2).$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

Euler's theorem for homogeneous functions:

A function $f(x,y)$ is said to be a homogeneous function of degree n in x and y if $f(tx,ty) = t^n f(x,y)$ for any positive t.

Euler's theorem:

If u is a homogeneous function of degree n in 'x' and y then $x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = nu$.

Problems:

1) If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then find $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Soln,

$$u(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

$$u(tx+ty+tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx} = t^0 u(x,y,z).$$

$\therefore$ u is a homogeneous function of degree 0.

$\therefore$ By Euler's thm,

$$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}+z\cdot\frac{\partial u}{\partial z}=nu=0\cdot u=0.$$

2. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\sin 2u$.

Soln:

27/12/22
civil : [illegible], 1, 3, 7, 8
Mech : 2, 9, 12, 15, 16, 20, 23, 26, 28

26/12/22
civil : 9, 4
Mech : 5, 24, [illegible], 8
Extra class.

Present Mech : 12.

Given, $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u=\frac{x^3+y^3}{x-y}$$

Let $f(x,y)=\tan u=\frac{x^3+y^3}{x-y}$

$$f(tx,ty)=\frac{t^3x^3+t^3y^3}{tx-ty}$$

$$=\frac{t^3}{t}\left(\frac{x^3+y^3}{x-y}\right)$$

$$=t^2 f(x,y)$$

$\therefore$ $\tan u$ is a homogeneous function of degree 2.

$\therefore$ By Euler's thm,

$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=nf.$$

$$x\cdot\frac{\partial}{\partial x}\tan u+y\frac{\partial}{\partial y}\tan u=2\tan u$$

$$x\sec^2u\frac{\partial u}{\partial x}+y\sec^2u\frac{\partial u}{\partial y}=2\tan u$$

$$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2u}$$

$$=\frac{2\sin u}{\cos u}\times\cos^2u$$

$$=\sin 2u$$

Scanned with OKEN Scanner

3. If $\cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, then prove that $x\frac{\partial^2 u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

Soln:

Let $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $f(x,y) = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t}{t^{1/2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$= t^{1-1/2} f(x,y)$$

$$= t^{1/2} f(x,y)$$

$\therefore$ f is a homogeneous function of degree $\frac{1}{2}$.

By Euler's thm, $x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = nf$.

$$x\cdot\frac{\partial}{\partial x}\cos u + y\cdot\frac{\partial}{\partial y}\cos u = \frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = \frac{-1}{2}\cot u.$$

4) If u is a homogeneous function of degree n in x & y, then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

Soln:

By Euler's thm, $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = nu$ ——(1).

Diff. (1) partially w.r.to 'x'.

$$x\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\cdot\frac{\partial^2 u}{\partial x\partial y} = n\cdot\frac{\partial u}{\partial x}.$$

$$x\cdot\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}.$$

$$= (n-1)\frac{\partial u}{\partial x} \quad \text{——(2)}.$$

Diff (1) partially w.r.to 'y'

$$x\cdot\frac{\partial^2 u}{\partial y\partial x} + y\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = x\cdot\frac{\partial u}{\partial y}.$$

$$x\cdot\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}=\cancel{x}\,n\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}$$

$$=(\cancel{xo})(n-1)\frac{\partial u}{\partial y} \qquad -(3).$$

$(2)x+(3)xy \Rightarrow$
$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=(n-1)x\frac{\partial u}{\partial x}+(n-1)y\frac{\partial u}{\partial y}$$

$$=(n-1)\left[x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=(n-1)\,nu. \quad [\text{by } 1]$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

5) If $u=x^2\tan^{-1}y/x-y^2\tan^{-1}x/y$. then find the value of $x^2\frac{\partial^2 u}{\partial x^2}+2xy\cdot\frac{\partial^2 u}{\partial x\partial y}+y^2\cdot\frac{\partial^2 u}{\partial y^2}$.

Soln:

Given, $u(x,y)=x^2\tan^{-1}y/x-y^2\tan^{-1}y/x.$

$$u(tx,ty)=t^2x^2\tan^{-1}\frac{tx}{ty}-t^2y^2\tan^{-1}\frac{ty}{tx}.$$

$$=t^2\left[x^2\tan^{-1}\frac{ty}{tx}-t^2y^2\tan^{-1}\frac{ty}{tx}\right.$$

$$=t^2\left[x^2\tan^{-1}y/x-y^2\tan^{-1}y/x\right]$$

$$=t^2\,u(x,y)$$

$\therefore$ u is a homogeneous function of degree 2.

$\therefore$ By Euler's thm,

$$x^2\cdot\frac{\partial^2 u}{\partial x^2}+2xy\cdot\frac{\partial^2 u}{\partial x\partial y}+y^2\cdot\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$$=2(2-1)u$$

$$=2u.$$

6) If $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that

i) $x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$

Soln:

Given $u = \sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}.$$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t}{t^{1/2}} f(x,y).$$

$$= t^{1/2} f(x,y)$$

$\therefore f = \sin u$ is a homogeneous function of degree $1/2$.

By Euler's thm, $x.\dfrac{\partial f}{\partial x} + y.\dfrac{\partial f}{\partial y} = nf.$

$$x.\frac{\partial}{\partial x}(\sin u) + y.\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$x.\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \quad \text{——— (1).}$$

(ii) Diff (1) partially w.r.to 'x'.

$$x.\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

$$x.\frac{\partial^2 u}{\partial x^2} + y.\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\sec^2 u.\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}.$$

$$x.\frac{\partial^2 u}{\partial x^2} + y.\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}.$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\frac{\partial u}{\partial x} \quad \text{——— (2).}$$

Diff (1) p.w.r.to 'y'.

$$x.\frac{\partial^2 u}{\partial y\partial x} + y.\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}.$$

$$x.\frac{\partial^2 u}{\partial y\partial x} + y.\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\frac{\partial u}{\partial y} \quad \text{——— (3).}$$

X'ing (2) by 'x', (3) by y and adding,

$$x^2.\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{2}\sec^2 u - 1\right)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= \left[\frac{1}{2\cos^2 u} - 1\right] \frac{1}{2} \tan u$$

$$= -\left[\frac{2\cos^2 u - 1}{2\cos^2 u}\right] \frac{1}{2} \frac{\sin u}{\cos u}$$

$$= \frac{-\sin u \cos 2u}{4\cos^3 u} \quad \left[\because 2\cos^2 u - 1 = \cos 2u\right].$$

Total Derivatives - Change of Variables - Partial Differentiation of implicit functions:

1. Find $\frac{dy}{dx}$ when $x^3 + y^3 = 3axy$.

Soln:

Let $f(x,y) = x^3 + y^3 - 3axy$.

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = -\frac{x^2 + ay}{y^2 - ax}.$$

2. If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, then find $\frac{du}{dx}$.

Soln:

Given $u = x\log(xy) = x[\log x + \log y]$

$$\frac{\partial u}{\partial x} = x\left[\frac{1}{x} + 0\right] + [\log x + \log y](1).$$

$$= 1 + \log x + \log y.$$

$$\frac{\partial u}{\partial y} = x(0 + 1/y) + (\log x + \log y)(0) = x/y.$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

$$= 1 + \log x + \log y + \frac{x}{y}\frac{dy}{dx} \quad \text{——} (1).$$

Given $x^3 + y^3 + 3xy = 1$.

Diff w.r.to 'x'.

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left[y(1) + x\frac{dy}{dx}\right] = 0$$

$$x^2 + y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$[y^2 + x]\frac{dy}{dx} = -[x^2 + y].$$

$$\frac{dy}{dx} = -\frac{x^2+y}{y^2+x}$$

$$\therefore (1) \Rightarrow \frac{du}{dx} = 1+\log x + \log y + \frac{x}{y}\left[-\left(\frac{x^2+y}{y^2+x}\right)\right]$$

$$= 1+\log x + \log y - x/y\left(\frac{x^2+y}{y^2+x}\right).$$

3. If $g(x,y) = \psi(u,v)$ where $u = x^2-y^2$ and $v = 2xy$, then prove that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right].$$

Soln:

Given $g(x,y) = \psi(u,v)$.

$u = x^2-y^2$ $\qquad$ $v = 2xy$

$\frac{\partial u}{\partial x} = -2y$ $\qquad$ $\frac{\partial v}{\partial y} = 2x.$

$\frac{\partial u}{\partial y} = -2y.$ $\qquad$ $\frac{\partial v}{\partial x} = 2y.$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x}.$$

$$= \frac{\partial \psi}{\partial u}(2x) + \frac{\partial \psi}{\partial v}(2y)$$

$$= 2x\frac{\partial \psi}{\partial u} + 2y\cdot\frac{\partial \psi}{\partial v}$$

$$\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$$

$$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v}(2x)$$

$$= -2y\frac{\partial \psi}{\partial u} - 2x\frac{\partial \psi}{\partial v}$$

$$\frac{\partial}{\partial y} = -2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}.$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial \psi}{\partial u} + 2y\cdot\frac{\partial \psi}{\partial v}\right).$$

$$= 4x^2\frac{\partial^2 \psi}{\partial u^2} + 4xy\frac{\partial^2 \psi}{\partial u \partial u} + 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4y^2\frac{\partial^2 \psi}{\partial v^2} \quad (1)$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \left(-2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial u}\right)\left(-2y\frac{\partial \psi}{\partial u} + 2x\frac{\partial \psi}{\partial v}\right)$$

$$= 4y^2\frac{\partial^2 \psi}{\partial u^2} + 4xy\frac{\partial^2 \psi}{\partial u \partial v} + 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4y^2\frac{\partial^2 \psi}{\partial v^2}$$

$$+ 4y^2\frac{\partial^2 \psi}{\partial u^2} - 4xy\frac{\partial^2 \psi}{\partial u \partial v} - 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4x^2\frac{\partial^2 \psi}{\partial v^2}$$

$$= 4(x^2+y^2)\frac{\partial^2 \psi}{\partial u^2} + 4(x^2+y^2)\frac{\partial^2 \psi}{\partial v^2}.$$

$= \cancel{4(x^2+y^2)}$

$$= 4(x^2+y^2)\left[\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right]$$

4. If $z = f(y-z, z-x, x-y)$, show that $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+\frac{\partial z}{\partial z}=0$.

Soln:

Let $u = y-z$, $v = z-x$, $w = x-y$.

$$z = f(u,v,w)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0)+\frac{\partial f}{\partial v}(-1)+\frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial w}(1)+\frac{\partial f}{\partial v}(0)+\frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u}-\frac{\partial f}{\partial w}.$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$= \frac{\partial f}{\partial u}(-1)\frac{\partial f}{\partial v}(1)+\frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}.$$

$$\therefore \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+\frac{\partial z}{\partial z} = -\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}+\frac{\partial f}{\partial u}-\frac{\partial f}{\partial w}-\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$$

$$= 0.$$

5. If z is a function of x and y where $x = e^u+e^{-v}$ and $y = e^{-u}-e^v$, show that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

Soln:

Given $z = f(x,y)$.

$x = e^u+e^{-v}$ | $y = e^{-u}-e^v$

$\frac{\partial x}{\partial u} = e^u$ | $\frac{\partial y}{\partial u} = -e^{-u}$

$\frac{\partial x}{\partial v} = -e^{-v}$ | $\frac{\partial y}{\partial v} = -e^v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^u+\frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^v).$$

$$= e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \qquad\qquad = -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}.$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\cdot\frac{\partial z}{\partial x} - e^{-u}\cdot\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$= (e^{u} + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^{v})\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

6. Transform the eqn $z_{xx} + 2z_{xy} + z_{yy} = 0$ by changing the independent variables using $u = x-y$ and $v = x+y$.

Soln:

$$u = x - y \qquad\qquad v = x + y$$

$$\frac{\partial u}{\partial x} = 1 \qquad\qquad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = -1 \qquad\qquad \frac{\partial v}{\partial y} = 1.$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad\qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 1 \qquad\qquad = \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}\cdot 1$$

$$= -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad\qquad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) \qquad\qquad \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \qquad\qquad = \left(-\partial/\partial u + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2} \qquad\qquad = -\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u\partial v} - \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \partial/\partial v\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 v}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v} - \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}.$$

$$z_{xx} + 2z_{xy} + z_{yy} = 0.$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} - 2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u}$$

$$+ 2\frac{\partial^2 z}{\partial v^2} + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$4\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} = 0$$

$$\Rightarrow 4Z_{vv} + 2Z_{uv} - 2Z_{vu} = 0$$

$$\Rightarrow 2Z_{vv} + Z_{uv} - Z_{vu} = 0.$$

JACOBIANS :

If u_1, u_2, u_3 are functions of two variables x_1, x_2, x_3

then $$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \frac{\partial u_1}{\partial x_3} \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \frac{\partial u_3}{\partial x_3} \end{vmatrix}.$$

Problems:

1. If $x = r\cos\theta$, $y = r\sin\theta$ find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Soln:

Given $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \frac{\partial x}{\partial \theta} \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

ii) $$\frac{\partial(x,y)}{\partial(r,\theta)} \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\Rightarrow r \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1 \Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = 1/r.$$

2) Find the Jacobian of y_1, y_2, y_3 w.r.to x_1, x_2, x_3, $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

Soln:

Given $y_1 = \frac{x_2 x_3}{x_1}$ $\quad y_2 = \frac{x_3 x_1}{x_2}$ $\quad y_3 = \frac{x_1 x_2}{x_3}$

$\frac{\partial y_1}{\partial x_1} = \frac{-x_2 x_3}{x_1}$ $\quad \frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$ $\quad \frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$

$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$ $\quad \frac{\partial y_2}{\partial x_2} = \frac{-x_3 x_1}{x_2^2}$ $\quad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$

$\frac{\partial y_2}{\partial x_3} = \frac{x_2}{x_1}$ $\quad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$ $\quad \frac{\partial y_3}{\partial x_3} = \frac{-x_1 x_2}{x_3^2}$.

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \partial y_2/\partial x_2 & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ x_3/x_2 & \frac{-x_3 x_1}{x_2^2} & x_1/x_2 \\ x_2/x_3 & x_1/x_3 & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\frac{x_1^2 x_3 x_2}{x_2^2 x_3^2} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left[\frac{-x_1 x_2 x_3}{x_2 x_3^2} - \frac{x_1 x_2}{x_2 x_3}\right]$$

$$+ \frac{x_2}{x_1}\left[\frac{x_1 x_3}{x_2 x_3} + \frac{x_3 x_1 x_2}{x_3 x_2^2}\right]$$

$$= \frac{-x_2 x_3 x_1^2 x_3 x_2}{x_1^2 x_2^2 x_3^2} + \frac{x_2 x_3 x_1^2}{x_1^2 x_2 x_3} + \frac{x_1 x_2 x_3^2}{x_1 x_2 x_3^2} + \frac{x_1 x_2 x_3}{x_1 x_2 x_3}$$

$$+ \frac{x_2 x_1 x_3}{x_1 x_2 x_3} + \frac{x_1 x_3 x_2^2}{x_1 x_3 x_2^2}.$$

$$= -1 + 1 + 1 + 1 + 1 + 1 = 4$$

3. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

Soln:

Given, $u = \frac{yz}{x}$ $\quad v = \frac{zx}{y}$ $\quad w = \frac{xy}{z}$

$$\frac{\partial u}{\partial x} = \frac{-yz}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{z}{y} \qquad \frac{\partial w}{\partial x} = y/z$$

$$\frac{\partial u}{\partial y} = \frac{z}{x} \qquad \frac{\partial v}{\partial y} = \frac{-zx}{y^2} \qquad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{x} \qquad \frac{\partial v}{\partial z} = x/y \qquad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \partial v/\partial y & \frac{\partial v}{\partial z} \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} -yz/x^2 & z/x & y/x \\ z/y & -zx/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{zx^2y}{z^2y^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{zy}\right] + \frac{y}{x}\left[\frac{xz}{yz} + \frac{zxy}{y^2z}\right]$$

$$= \frac{-x^2y^2z^2}{x^2y^2z^2} + \frac{x^2yz}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} + \frac{xyz}{xyz} + \frac{xy^2z}{xy^2z}$$

$$= -1+1+1+1+1+1 = 4.$$

4. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Soln:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\sin\theta\cos\phi$ $\qquad y = r\sin\theta\sin\phi$ $\qquad z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = +r\cos\theta\cos\phi \qquad \partial y/\partial\theta = r\cos\theta\sin\phi \qquad \partial z/\partial\theta = -\sin\theta$$

$$\partial x/\partial\phi = -r\sin\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \partial z/\partial\phi = 0$$

Scanned with OKEN Scanner

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Expand using third row,

$$= \cos\theta\,[r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi]$$
$$+ r\sin\theta\,[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi]$$

$$= \cos\theta\,[r^2\cos\theta\sin\theta(\cos^2\phi + \sin^2\phi)]$$
$$+ r\sin\theta\,[r\sin^2\theta(\cos^2\phi + \sin^2\phi)]$$

$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta\,(\cos^2\theta + \sin^2\theta)$$

$$= r^2\sin\theta.$$

Taylor's Series for Function of two Variables:

Formula:

$$f(x,y) = f(a,b) + \frac{1}{1!}[h f_x(a,b) + k f_y(a,b)]$$
$$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b)$$
$$+ k^3 f_{yyy}(a,b)] + \cdots, \text{ where } h = x-a,\ k = y-b$$

Problems:

1. Expand $e^x\cos y$ about $(0, \pi/2)$ up to third degree terms using taylor's series:

Soln:

Function	Value at $(0, \pi/2)$.
$f(x,y) = e^x\cos y$	$f(x, \pi/2) = e^0\cos\pi/2 = 0$
$f_x = e^x\cos y$	$f_x(0,\pi/2) = e^0\cos\pi/2 = 0$
$f_y = -e^x\sin y$	$-e^0\sin\pi/2 = -1$

$f_{xx} = e^x \cos y$ $\qquad e^0 \cos \pi/2 = 0$

$f_{xy} = -e^x \sin y$ $\qquad -e^0 \sin \pi/2 = -1$

$f_{yy} = -e^x \cos y$ $\qquad -e^0 \cos \pi/2 = 0$

$f_{xxx} = e^x \cos y$ $\qquad f_{xxx}(0, \pi/2) = e^x \cos \pi/2 = 0$

$f_{xxy} = -e^x \sin y$ $\qquad f_{xxy} = -e^0 \sin \pi/2 = -1$

$f_{xyy} = -e^x \cos y$ $\qquad -e^0 \cos \pi/2 = 0$

$f_{yyy} = e^x \sin y$ $\qquad e^0 \sin \pi/2 = 1.$

Substitute all values in Taylors series

$$f(x,y) = 0 + [x(0) + (y-\pi/2)(-1)] + \frac{1}{2!}[x^2(0) + 2x(y-\pi/2)(-1) + (y-\pi/2)^2(0)] + \frac{1}{3!}[x^3(0) + 3x^2(y-\pi/2)(-1) + 3x(y-\pi/2)^2(0) + (y-\pi/2)^3(+1)]$$

$$= -y + \pi/2 - 1/2(2x(y-\pi/2)) + \frac{1}{6}[-3x^2(y-\pi/2) + (y-\pi/2)^3]$$

2. Expand $\sin xy$ in powers of $x-1$ and $y-\pi/2$ upto second degree terms by using Taylor's series.

Soln:

Function $\qquad$ value at $(1, \pi/2)$

$f(x,y) = \sin(xy)$ $\qquad f(1, \pi/2) = \sin \pi/2 = 1$

$f_x = y\cos(xy)$ $\qquad \pi/2 \cos \pi/2 = 0$

$f_y = x\cos(xy)$ $\qquad \cos \pi/2 = 0$

$f_{xx} = -y^2 \sin(xy)$ $\qquad -\pi^2/4 \sin \pi/2 = -\pi^2/4$

$f_{xy} = y(-x\sin(xy)) + \cos(xy)$ $\qquad -\pi/2 \sin \pi/2 + \cos \pi/2 = -\pi/2$

$f_{yy} = -x^2 \sin xy$ $\qquad -\sin \pi/2 = -1.$

Taylor's series:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$

Here $a=1$, $b=\pi/2$

$$h = x-a = x-1, \quad k = y-b = y-\pi/2$$

$$f(x,y) = 1 + (x-1)(0) + (y-\pi/2)(0) + \frac{1}{2!}\left[(1-1)^2\left(-\pi^2/4\right) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)\right]$$

$$= 1 + 1/2\left[\frac{-\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right].$$

3. Expand $e^x \sin y$ in powers of x & y upto third degree terms using Taylor's series.

Soln:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b)\right.$$

Function	Value at $(0,0)$.
$f(x,y) = e^x \sin y$	$f(0,0) = e^0 \sin 0 = 0$.
$f_x = e^x \sin y$	$f_x = 0$
$f_y = e^x \cos y$	$f_y = 1$.
$f_{xx} = e^x \sin y$	$f_{xx} = 0$
$f_{xy} = e^x \cos y$	$f_{xy} = 1$
$f_{yy} = -e^x \sin y$	$f_{yy} = 0$
$f_{xxx} = e^x \sin y$	$f_{xxx} = 0$
$f_{xxy} = e^x \cos y$	$f_{xxy} = 1$
$f_{xyy} = -e^x \sin y$	$f_{xyy} = 0$
$f_{yyy} = -e^x \cos y$	$f_{yyy} = -1$

Here $a=0$, $b=0$

$h = x-a = x$

$k = y-b = y$.

$\therefore f(x,y) = 0 + \frac{1}{1!}[x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)]$

$+ \frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)]$

$= y + \frac{1}{2}(2xy) + \frac{1}{6}3x^2y + \frac{1}{6}(-y^3)$

$= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3$.

Maxima and Minima of functions of two variables:

Necessary conditions. for a maximum or minimum $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Sufficient conditions: If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$, then

i) $f(a,b)$ is maximum value of $AC - B^2 > 0$ and $A < 0$ (or $B < 0$).

ii) $f(a,b)$ is minimum value if $AC - B^2 > 0$ and $A > 0$ (or $B > 0$)

iii) $f(a,b)$ is not extremum (saddle) if $AC - B^2 < 0$

iv) If $AC - B^2 = 0$, then the test is inconclusive.

Stationary Value:

1. Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Soln:

Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

$A = f_{xx} = 6x$

$B = f_{xy} = 0$

$C = f_{yy} = 6y$

$AC - B^2 = (6x)(6y) - 0 = 36xy$.

To find stationary points,

$f_x = 0$ $\qquad$ $f_y = 0$

$3x^2 - 3 = 0$ $\qquad$ $3y^2 - 12 = 0$

$3(x^2-1) = 0$ $\qquad$ $3(y^2-4) = 0$

$x^2 - 1 = 0$ $\qquad$ $y^2 - 4 = 0$

$x^2 = 1$ $\qquad$ $y^2 = 4$

$x = \pm 1$ $\qquad$ $y = \pm 2$.

$\therefore$ The stationary points are,

$(1,2), (1,-2), (-1,2), (-1,-2)$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6>0$	$6>0$	$-6>0$	$-6<0$
$B = 0$	0	0	0	0
$AC-B^2 = 36xy$	$72>0$	$-72<0$	$-72<0$	$72>0$
conclusion	minimum	saddle	saddle	maximum.

$\therefore$ Maximum value of $f(x,y)$ is

$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$

$= -1 - 8 + 3 + 24 + 20$

$= 38$.

minimum value of $f(x,y)$ is

$f(1,2) = 1^3 + (2)^3 - 3(1) - 12(2) + 20$

$= 1 + 8 - 3 - 24 + 20$

$= 2$.

2) Find the extreme value of $f(x,y) = x^3y^2(1-x-y)$

Soln:

$f(x,y) = x^3y^2(1-x-y)$

$= x^3y^2 - x^4y^2 - x^3y^3$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$

$$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$$

$$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$$

To find stationary points,

$f_x = 0$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(3 - 4x - 3y) = 0$$

$$\Rightarrow x = 0, y = 0, 4x + 3y = 3 \quad \text{--- (1)}$$

$f_y = 0$

$$2x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(2 - 2x - 3y) = 0$$

$$\Rightarrow x = 0, y = 0, 2x + 3y = 2 \quad \text{--- (2)}$$

$$(1) \Rightarrow 4x + 3y = 3$$

$$(2) \Rightarrow \overset{(-)}{2x} \overset{(-)}{+ \cancel{3y}} = \overset{(-)}{2}$$

$$2x = 1$$

$$\boxed{x = 1/2}$$

subs $x = 1/2$ in (2).

$$2\frac{1}{2} + 3y = 2$$

$$1 + 3y = 2$$

$$y = 1/3$$

$\therefore$ The stationary points are, $(0,0)$ and $(1/2, 1/3)$.

At $(1/2, 1/3)$.

$$A = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3$$

$$= 6 \times 1/2 \times 1/9 - 12 \times 1/4 \times 1/9 - 6 \times 1/2 \times 1/27$$

$$= 1/3 - 1/3 - 1/9 = -1/9$$

$$B = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^2$$

$$= 6 \times 1/4 \times 1/3 - 8 \times 1/8 \times 1/3 - 9 \times 1/4 \times 1/9.$$

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{6-4-3}{12} = -\frac{1}{12}.$

$C = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$

$= 2 \times \frac{1}{8} - 2 \times \frac{1}{16} - 6 \times \frac{1}{8} \times \frac{1}{3}.$

$= \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = \frac{-1}{8}$

$AC - B^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

$A = -\frac{1}{9} < 0$

$\therefore f\left(\frac{1}{2}, \frac{1}{3}\right)$ is maximum

maximum value of $f(x,y)$ is $f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$

$= \frac{1}{8} \times \frac{1}{9}\left[\frac{6-3-2}{6}\right]$

$= \frac{1}{72} \times \frac{1}{6}$

$= \frac{1}{432}$

5. Find the maximum or minimum values of $f(x,y) = 3x^2 - y^2 + x^3$

Soln:

Given $f(x,y) = 3x^2 - y^2 + x^3$

$f_x = 6x + 3x^2$

$f_y = -2y$

$A = f_{xx} = 6 + 6x$

$B = f_{xy} = 0$

$C = f_{yy} = -2.$

To find stationary points:

$f_x = 0$	$f_y = 0$
$6x + 3x^2 = 0$	$-2y = 0$
$3x(2+x) = 0$	$y = 0$
$x = 0, -2$	

$\therefore$ The stationary points are $(0,0)$ & $(-2,0)$.

	$(0,0)$	$(-2,0)$
$A = 6+6x$	$6>0$	$-6<0$
$B=0$	0	0
$C=-2$	-2	-2
$AC-B^2$	$-12<0$	$12>0$
Conclusion	saddle	maximum

$\therefore$ maximum value of $f(x,y)$ is

$$f(-2,0) = 3(-2)^2 - 0 + (-2)^3$$
$$= 3(4) - 8 = 12-8 = 4.$$

4. Examine the maxima and minima of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72$

Soln:

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x.$$
$$f_x = 3x^2 + 3y^2 - 30x + 72$$
$$f_y = 6xy - 30y$$
$$A = f_{xx} = 6x - 30$$
$$B = f_{xy} = 6y$$
$$C = f_{yy} = 6y - 30$$

To find the stationary points

$f_x = 0$	$f_y = 0$
$3x^2+3y^2-30x+72=0$	$6xy-30y=0$
	$6y(x-5)=0$
	$y=0,\ x=5$

when $y=0$,

$$3x^2 - 30x + 72 = 0$$
$$3(x^2 - 10x + 24) = 0$$
$$(x-6)(x-4) = 0$$
$$x = 6, 4$$

$\therefore$ The stationary points are $(6,0), (4,0)$.

	$(6,0)$	$(4,0)$
$A = 6x - 30$	$36 - 30 = 6 > 0$	$24 - 30 = -6 < 0$
$B = 6y$	0	0
$C = 6y - 30$	-30	-30
$AC - B^2$	$-180 < 0$	$180 > 0$
Conclusion	saddle	maximum

$\therefore$ Maximum value of $f(x,y)$ is

$$f(4,0) = (4)^3 + 0 - 15(4)^2 - 0 + 72(4)$$
$$= 64 - 240 + 288$$
$$= 328.$$

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:

To find the maximum and minimum value of $f(x,y,z)$ where x,y,z are subject to a constraint equation $g(x,y,z)=0$ we define the function,

$$F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) \quad \text{---(1)}$$

where λ is called Lagrange's multiplier which is independent of x,y,z.

The necessary conditions for a maximum or minimum are $\frac{\partial F}{\partial x} = 0$ ---(2), $\frac{\partial F}{\partial y} = 0$ ---(3) and $\frac{\partial F}{\partial z} = 0$ ---(4).

PROBLEMS:

1. A rectangular box opens at the top is to have a volume of 32 cc. Find the dimensions of the box that requires the least material for its construction.

Soln:

Let x, y, z be the length, breadth and height of the box.

Surface area $= xy + 2yz + 2xz = f(x,y,z)$.

$$\text{Volume} = xyz = 32 = g(x,y,z).$$

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z).$$

$$\Rightarrow F(x,y,z) = xy + 2yz + 2xz + \lambda(xyz - 32) \quad \text{—— (1)}.$$

$$\frac{\partial F}{\partial x} = 0$$

$$y + 2z + \lambda yz = 0$$

$$y + 2z = -\lambda yz$$

$$\frac{y+2z}{yz} = -\lambda$$

$$\frac{1}{z} + \frac{2}{y} = -\lambda \quad \text{(2)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + \lambda xz = 0$$

$$x + 2z = -\lambda xz$$

$$\frac{x+2z}{xz} = -\lambda$$

$$\frac{1}{z} + \frac{2}{x} = -\lambda \quad \text{(3)}$$

$$\frac{\partial F}{\partial z} = 0$$

$$2y + 2x + \lambda xy = 0$$

$$2y + 2x = -\lambda xy$$

$$\frac{2y+2x}{xy} = -\lambda$$

$$\frac{2}{x} + \frac{2}{y} = -\lambda \quad \text{—— (4)}.$$

From (2) & (3)

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$x = y \quad \text{—— (5)}$$

From (3) & (4)

$$\frac{1}{z} + \cancel{\frac{2}{x}} = \cancel{\frac{2}{x}} + \frac{2}{y}$$

$$\frac{1}{z} = \frac{2}{y}$$

$$y = 2z \quad \text{—— (6)}$$

From (5) & (6), $x = y = 2z$

Volume $xyz = 32$

$$(2z)(2z)z = 32$$

$$4z^3 = 32$$

$$z^3 = \frac{32}{4} = 8$$

$$z = 2$$

$\therefore x = 4,\ y = 4,\ z = 2$

$\therefore$ Dimensions of the box are 4, 4, 2.

2) The temperature $u(x,y,z)$ at any point in space is $u = 400xyz^2$. Find the highest temperature on surface of the sphere $x^2+y^2+z^2=1$

Soln:

Given $u = f = 400xyz^2$

$g = x^2+y^2+z^2-1$

$F(x,y,z,\lambda) = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$ —— (1).

$F_x = 0$

$400yz^2 + \lambda(2x) = 0$

$400yz^2 = -2\lambda x$

$\frac{400yz^2}{2x} = -\lambda$

$\frac{200yz^2}{x} = -\lambda$ —— (2)

$F_y = 0$

$400xz^2 + \lambda(2y) = 0$

$400xz^2 = -\lambda 2y$

$\frac{400xz^2}{2y} = -\lambda$

$\frac{200xz^2}{y} = -\lambda$ —— (3)

$F_z = 0$

$800xyz + \lambda(2z) = 0$

$800xyz = -\lambda 2z$

$\frac{800xyz}{2z} = -\lambda$

$400xy = -\lambda$ —— (4)

From (2) & (3)

$\frac{200yz^2}{x} = \frac{200xz^2}{y}$

$\frac{y}{x} = \frac{x}{y}$

$y^2 = x^2$ —— (5)

From (3) & (4)

$\frac{200xz^2}{y} = 400xy$

$\frac{z^2}{y} = 2y$

$2y^2 = z^2$ —— (6).

From (5) & (6).

$x^2 = y^2 = \frac{1}{2}z^2$

we have $x^2+y^2+z^2 = 1$

$\frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 = 1$

$\frac{z^2+z^2+2z^2}{2} = 1$

$\frac{4z^2}{2} = 1 \Rightarrow 2z^2 = 1 \Rightarrow z^2 = \frac{1}{2} \Rightarrow z = \pm\frac{1}{\sqrt{2}}$

$\therefore x^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

and $y^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm 1/2$.

$\therefore$ Temperature $u = 400xyz^2$

$= 400(1/2)(1/2)(1/\sqrt{2})^2$

$= 400 \times \frac{1}{4} \times \frac{1}{2} = 50$

$\therefore$ The maximum temperature is 50.

3. Find the maximum volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Soln:

Let the vertex of the parallelopiped by (x, y, z).

All other vertices will be $(\pm x, \pm y, \pm z)$

sides of the solid be $2x, 2y, 2z$.

Volume $V = (2x)(2y)(2z) = 8xyz = f$.

we have to maximize V subject to the condition

$$g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$F(x,y,z) = 8xy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) \quad \text{--- (1)}$$

$F_x = 0$

$$8yz + \frac{2xz}{a^2} = 0$$

$$8yz = \frac{x}{a^2}$$

$$\frac{-4yz}{\lambda} = \frac{x}{a^2}$$

$$\frac{-4xyz}{\lambda} = \frac{x^2}{a^2} \quad \text{--- (2)}$$

$F_y = 0$

$$8xz + \frac{2y\lambda}{b^2} = 0$$

$$8xz = -\frac{2y\lambda}{b^2}$$

$$\frac{8xz}{-2\lambda} = \frac{y}{b^2}$$

$$\frac{-4xzy}{\lambda} = \frac{y^2}{b^2} \quad \text{--- (3)}$$

$F_z = 0$

$$8xy + \frac{2z\lambda}{c^2} = 0$$

$$8xy = -\frac{2z\lambda}{c^2}$$

$$\frac{8xy}{-2\lambda} = \frac{z}{c^2}$$

$$\frac{-4xy}{\lambda} = \frac{z}{c^2}$$

$$\frac{-4xyz}{\lambda} = \frac{z^2}{c^2} \quad \text{--- (4)}$$

From (2), (3) & (4)

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\frac{x^2}{a^2}+\frac{x^2}{a^2}+\frac{x^2}{a^2}=1$$

$$\frac{3x^2}{a^2}=1$$

$$x^2=\frac{a^2}{3} \Rightarrow x=\frac{a}{\sqrt{3}}$$

Similarly, $y=\frac{b}{\sqrt{3}}$, $z=\frac{c}{\sqrt{3}}$

$\therefore$ Extremum point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$, maximum point.

$\therefore$ Maximum ~~point~~ value is $V=\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$

$$= \frac{8abc}{3\sqrt{3}}.$$

4) Find the minimum values of x^2yz^3 subjected to the condition $2x+y+3z=a$.

Soln:

Let $f = x^2yz^3$

$g = 2x+y+3z-a$

$F(x,y,z,\lambda) = x^2yz^3+\lambda(2x+y+3z-a)$ —— (1)

$F_x=0$	$F_y=0$	$F_z=0$
$2xyz^3+2\lambda=0$	$x^2z^3+\lambda=0$	$3x^2yz^2+3\lambda=0$
$2xyz^3=-2\lambda$	$x^2z^3=-\lambda$ —(3)	$3x^2yz^2=-3\lambda$
$xyz^3=-\lambda$ —(2)		$x^2yz^2=-\lambda$ —— (4).

From (2) & (3),	From (3) & (4)
$xyz^3=x^2z^3$	$x^2z^3=x^2yz^2$
$y=z$ —(5)	$z=y$ —(6)

From (5) & (6), $x=y=z$.

Given :- $2x+y+3z=a$

$2z+z+3z=a$

$6z=a$

$z = a/6$

$\therefore x = y = \frac{a}{6}$

$\therefore$ The stationary point is $\left(\frac{a}{6}, a/6, a/6\right)$

$\therefore$ minimum value of f is $(a/6)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3 = \frac{a^6}{6^6} = \left(\frac{a}{6}\right)^6$.

CHAPTER - 4

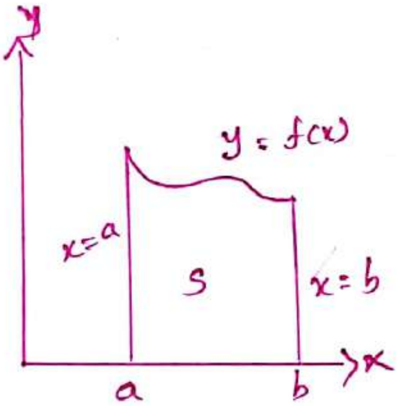
INTEGRAL CALCULUS

Introduction:

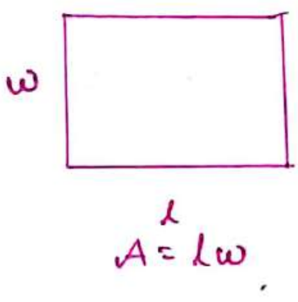
In Mathematics integral assign numbers to functions in a way that can describe dispacement, area, volume and open concepts that arise by continuing infinitial data.

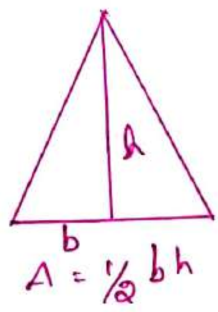
First let us concentrate to solve the area problem.

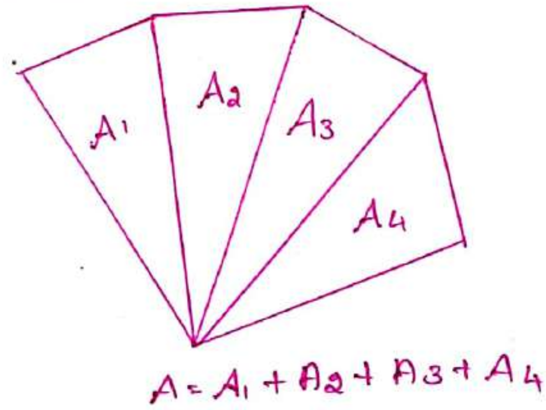
Given a function f which is continuous and non-negative on an interval $[a, b]$, find the areas between the graph of 'f' and the interval $[a, b]$ on the x-axis.



From this diagram, S is bounded by the graph of a continuous function f (where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$ and the x-axis.





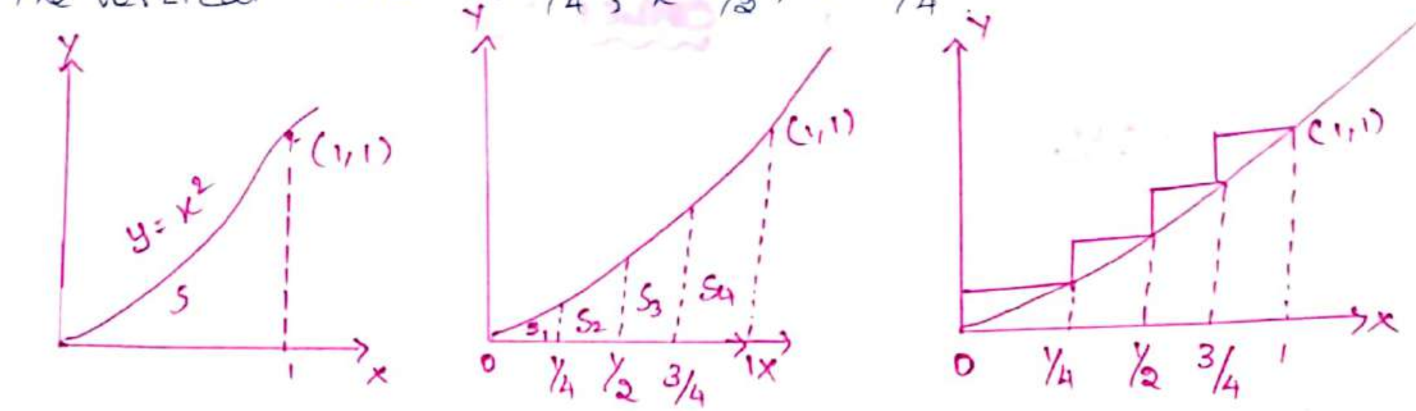


The area problem:

1) Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Soln:

Given that the area S is between 0 and 1 we divide S into 4 strips S_1, S_2, S_3 and S_4 by drawing the vertical lines $x = 1/4$, $x = 1/2$, $x = 3/4$



We can approximate each strip by a rectangle that has the same base as the strip and whose weight is the same as the right edge of the strip.

The weight of these rectangles are the values of the function $f(x) = x^2$ at the right end pts of the subintervals, $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ & $[3/4, 1]$.

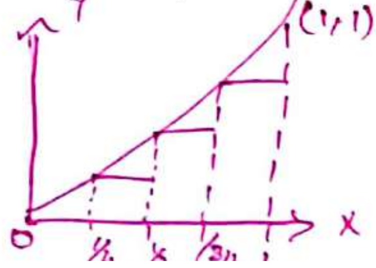
Each rectangle has width $1/4$ and the height are $(1/4)^2, (1/2)^2, (3/4)^2$ and 1. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\left(1/4\right)^2 + \frac{1}{4}\left(1/2\right)^2 + \frac{1}{4}\left(3/4\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32} = 0.46875$$

$\therefore$ From the above diagram we see that the area A of S is less than R_4.

$\therefore A < 0.46875$.

Instead of using the above rectangles, we can use the smaller rectangles from the following graph.



From the graph the weights are the values of f at the left end points of the subintervals. The sum of the areas of these approximating rectangle is

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(3/4\right)^2$$

$= 7/32 = 0.21875$.

we see that the area of S is larger than L_4, so we have lower and upper estimates for A.

$\therefore 0.21875 < A < 0.46875$.

we should repeat this procedure with a larger no. of strips. Now the given region is subdivided into 8 strips, of equal width.

$$L_8 = \frac{1}{8}(0)^2 + \frac{1}{8}(1/8)^2 + \frac{1}{8}\left(\frac{2}{8}\right)^2 + \frac{1}{8}(3/8)^2 + \frac{1}{8}(4/8)^2 + \frac{1}{8}(5/8)^2 + \frac{1}{8}(6/8)^2 + \frac{1}{8}(7/8)^2$$

$$= \frac{1}{8}\left[\frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64} + \frac{25}{64} + \frac{36}{64} + \frac{49}{64}\right]$$

$= 0.273434$

$$R_8 = \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}(2/8)^2 + \frac{1}{8}(3/8)^2 + 1/8(4/8)^2 + \frac{1}{8}(5/8)^2 + \frac{1}{8}(6/8)^2 + \frac{1}{8}(7/8)^2 + 1/8(1)^2$$

$R_8 = 0.3984375$

$\therefore 0.2734345 < A < 0.3984375$.

n	L_n	R_n
10	0.285000	0.385000
20	0.3087500	0.3587500
30	0.3168579	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

we can obtain better estimates by increasing the no. of strips.

$\therefore$ A good estimate is obtained by averaging these numbers.

$A = 0.333335$.

2) For the region S in $y = x^2$ from 0 to 1, show that the sum of the areas of the upper approximating rectangles approaches $1/3$.

i.e. $\lim_{n\to\infty} R_n = 1/3$.

Soln:

Let R_n be the sum of the areas of the n rectangles.

Each rectangle has width $\frac{1}{n}$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots n/n$. i.e, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$.

Then $R_n = \frac{1}{n}(1/n)^2 + \frac{1}{n}(2/n)^2 + \cdots + \frac{1}{n}(n/n)^2$

$$= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$R_n = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{1}{6} \frac{n\left(1+\frac{1}{n}\right)}{n} \cdot \frac{n \cdot (2+1/n)}{n}$$

$$\therefore \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{6}\left(1+1/n\right)\left(2+1/n\right) = \frac{1}{6} \cdot 1 \cdot 2 = 1/3$$

i.e, The sum of the areas of the upper approximating rectangles approaches $1/3$.

Definition:

The area of a region S that lies under the graph of the continuous function 'f' is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x]$$

we can get the same values for left end points.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + f(x_3)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

$$\therefore A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

3. Find the area under the curve $y = x^3$ on the interval $[0,1]$.

Soln:

Dividing $[0,1]$ into n strips of equal length.

$$\Delta x = \frac{1-0}{n} = 1/n$$

$$x_0 = 0,\ x_1 = 0 + 1/n = 1/n,\ x_2 = 0 + 2 \cdot 1/n = 2/n, \dots, x_n = 1.$$

If R_n is the right end point approximation using n approximating rectangles, then $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$.

$$A = \lim_{n\to\infty}\left[(1/n)^3(1/n) + (2/n)^3(1/n) + (3/n)^3(1/n) + \dots + (n/n)^3(1/n)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(1^3 + 2^3 + \dots + n^3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{1}{4}\lim_{n\to\infty}\left(1 + 1/n\right)^2$$

$$= \frac{1}{4}$$

Similarly, if L_n is the ~~right~~ left end point approximation using n approximating rectangles, then

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \lim_{n\to\infty}\left[(0/n)^3(1/n) + (1/n)^3(1/n) + (2/n)^3(1/n) + \dots + \left(\frac{n-1}{n}\right)^3(1/n)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(1^3 + 2^3 + 3^3 + \dots + (n-1)^3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n-1)}{2}\right)^2$$

$$= \frac{1}{4}\lim_{n\to\infty} \frac{n^4(1 - 1/n)^2}{n^4}$$

$$A = 1/4$$

The Definite Integral:

The limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x\right]$$

If 'f' is a function defined for $a \le x \le b$, we divide $[a,b]$ into n subintervals and of equal width $\Delta x = \frac{b-a}{n}$.

Let $x_0 = a, x_1, x_2, \dots, x_n = b$ be the end pts of the sub intervals and let $x_1^*, x_2^*, \dots, x_n^*$ be any sample pts in these subintervals, so x_i^* lies in the i^{th} subintervals $[x_{i-1}, x_i]$.

Then the definite integral of 'f' from a to b is given by,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\Delta x.$$

Provided that this limit exists and gives the same value for all possible points. If it exists, then f is integrable on $[a,b]$.

Theorem: 1

If 'f' is continuous on $[a,b]$ or if f has only a finite no. of discontinuities, then f is integrable on (a,b).

ie, the definite integral $\int_a^b f(x)dx$ exists.

Theorem: 2

If 'f' is integrable on $[a,b]$, then $\int_a^b$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Properties of definite integrals:

Consider the integral $\int_a^b f(x)dx$.

Let $a < b$. Then $\Delta x = \frac{a-b}{h}$.

$$\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

If $a=b$, then $\Delta x = 0$ & $\int_a^b f(x)\,dx = 0$.

Let us assume that f and g are continuous functions

Then (i) $\int_a^b c\,dx = c(b-a)$, where c is any constant.

ii) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

iii) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.

iv) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

v) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

vi) If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$.

vii) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

viii) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Pbm:

Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x=0$ and $x=2$, (a) using right end points Find an expression for A as a limit. (b) Estimate the area by taking the sample pts to be mid-points and using four subintervals and then ten sub-intervals.

Soln:

a) Since $a=0$ and $b=2$ then $\Delta x = \frac{2-0}{n} = 2/n$

$\therefore x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n, \ldots, x_i = \frac{2i}{n}$ & $x_n = \frac{2n}{n}$.

$\therefore R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$.

$$= e^{-2/n}(2/n) + e^{-4/n}(2/n) + \cdots + e^{-2n/n}(2/n)$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left[e^{-2/n} + e^{-4/n} + \cdots + e^{-2n/n}\right]$$

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

b). when $n=4$, $\Delta x = 0.5$. The sub-intervals are $[0, 0.5]$, $[0.5, 1.0]$, $[1.0, 1.5]$ and $[1.5, 2]$

The mid-pts are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$ and $x_4^* = 1.75$.

$$M_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x.$$

$$= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$$

$$= \frac{1}{2}\left(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}\right)$$

$M_4 = 0.8557$

when $x = 10$, the sub-intervals are $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 1.0]$, $[1.0, 1.2]$, $[1.2, 1.4]$, $[1.4, 1.6]$, $[1.6, 1.8]$, $[1.8, 2.0]$

The mid pts are $0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5, 1.7$ and 1.9.

$$A = M_{10} = f(0.1)\Delta x + f(0.3)\Delta x + f(0.5)\Delta x + f(0.7)\Delta x + \cdots + f(1.9)\Delta x.$$

$$= 0.2\left[e^{-0.1} + e^{-0.3} + e^{-0.5} + e^{-0.7} + e^{-0.9} + e^{-1.1} + e^{-1.3} + e^{-1.5} + e^{-1.7} + e^{-1.9}\right]$$

$A = 0.8632$

Pbm : 5.

Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample pts to be right end pts and $a=0, b=3$ and $n=6$.

Also Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Soln:

when $n = 6$, $\Delta x = \frac{b-a}{n} = \frac{1}{2}$

$\therefore$ The right end pts are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$

$x_5 = 2.5$ and $x_6 = 3.0$.

The Riemann sum is,

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x.$$

$$= f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x + f(2.0)\Delta x + f(2.5)\Delta x + f(3.0)\Delta x.$$

$$= \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9]$$

$$R_6 = -3.9375.$$

b) with n sub-intervals, we have $\Delta x = \frac{b-a}{n} = 3/n$.

$\Rightarrow x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$ & in general $x_i = \frac{3i}{n}$.

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(3i/n)\, 3/n$$

$$= \lim_{n\to\infty} 3/n \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6(3i/n)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}(1+1/n)^2 - 27(1+1/n)\right]$$

$= \frac{81}{4} - 27$ $\left\{ \because \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 . \; \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right\}$

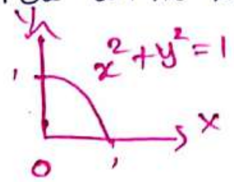
$\therefore \int_0^3 (x^3 - 6x)\, dx = -6.75$

6) Evaluate the following integrals by interpreting each in terms of areas (a) $\int_0^1 \sqrt{1-x^2}\, dx$ b) $\int_0^3 (x-1)\, dx$

Soln:

Let $f(x) = \sqrt{1-x^2}$. This integral as the area under the curve $y = \sqrt{1-x^2}$ from 0 to 1.

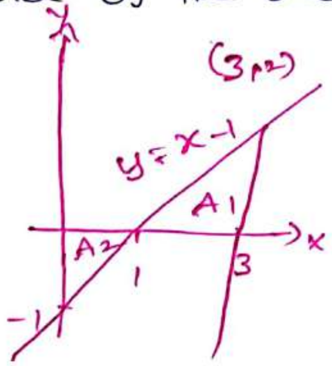
Since $y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$. [ie a quadratic circle with radius 1].



$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1) = \pi/4$

b) The graph of $y = 1-x$ is the line with slope 1.

we compute the integral as the difference of the areas of the two triangles.



$\int_0^3 (x-1)dx = A_1 - A_2$

$= \frac{1}{2}(2\times 2) - \frac{1}{2}(1\times 1)$

$\int_0^3 (x-1)dx = 1.5$

7) Prove that (a) $\int_a^b x\, dx = \frac{b^2-a^2}{2}$ and (b) $\int_a^b x^2\, dx = \frac{b^3-a^3}{3}$

Soln:

a) with n sub-intervals, we have $\Delta x = \frac{b-a}{n}$.

$\Rightarrow x_i = a + \frac{(b-a)}{n} i$. To evaluate the integral, we use Riemann sum

$\int_a^b x\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\left[a + \frac{(b-a)}{n} i\right]$

$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left[a + \left(\frac{b-a}{n}\right) i\right]$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n} a + \sum_{i=1}^{n}\left(\frac{b-a}{n}\right) i\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{b-a}{n}\right) a \sum_{i=1}^{n} 1 + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[a\left(\frac{b-a}{n}\right).n + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[a(b-a) + \frac{(b-a)^2}{2}(1+1/n)\right]$$

$$= ab - a^2 + \frac{1}{2}(a^2 - 2ab + b^2).1$$

$$= \frac{1}{2}[2ab - 2a^2 + a^2 - 2ab + b^2]$$

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}.$$

b) With n sub-intervals, we have $\Delta x = \frac{b-a}{n} \Rightarrow x_i = a + \left(\frac{b-a}{n}\right) i$

$$\therefore \int_a^b x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{b-a}{n}\right)\left[a + \left(\frac{b-a}{n}\right) i\right]^2$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n}\left[a + \left(\frac{b-a}{n}\right) i\right]^2$$

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left[a^2 + 2a\left(\frac{b-a}{n}\right) i + \left(\frac{b-a}{n}\right)^2 i^2\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{a^2(b-a)}{n} + 2a\left(\frac{b-a}{n}\right)^2 i + \left(\frac{b-a}{n}\right)^3 i^2\right]$$

$$\int_a^b x^2 dx = \lim_{n\to\infty}\left[\frac{a^2(b-a)}{n}\sum_{i=1}^{n} 1 + 2a\left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} i^2\right]$$

$$= a^2(b-a) + 2a(b-a)^2 \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2} + (b-a)^3 \lim_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$= a^2 b - a^3 + ab^2 - 2a^2 b - a^3 + b^3/3 - ab^2 + a^2 b - a^3/3$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3}.$$

2) Evaluate the integral by interpreting it in terms of axes

$\int_0^{10} |x-5|\, dx$.

Soln:

Let $f(x) = |x-5|$ between 0 & 10.

The value of the integral can be interpreted as the sum of the areas of the two triangles of base length 5 and height 5.

$\int_0^{10} |x-5|\, dx = 2\left(\frac{1}{2}\right) \cdot 5 \cdot 5 = 25$.

Theorem 1: [Fundamental theorem of calculus part-1]

If f is continuous on $[a,b]$ then the function g is defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

Theorem 2 [Fundamental theorem of calculus part-2]

If f' is continuous on $[a,b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any anti-derivative of f, i.e.) a function such that $F' = f$.

Problems:

1) Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\, dt$.

Soln:

Since $f(t) = \sqrt{1+t^2}$ is continuous, by part-1 of the fundamental theorem of calculus gives $g'(x) = \sqrt{1+x^2}$

2) Evaluate the integral $\int_1^3 e^x\, dx$.

Soln:

Since $f(x) = e^x$ is continuous everywhere.

w.k.t anti-derivative of $f(x)$ is $F(x) = e^x$

So part 2 of fundamental theorem gives, $\int_1^3 e^x\,dx = F(3) - F(1)$
$= e^3 - e^1$.

3) what is wrong with the following calculation

$$\int_{-1}^{3} (1/x^2)\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{3} = -4/3.$$

Soln:

Since $f(x) = \frac{1}{x^2} \geq 0$, $\int_a^b f(x)\,dx \geq 0$ when $f(x) \geq 0$.

Also $1/x^2$ is discontinuous on $[-1, 3]$, $f(x) = 1/x^2$ has an infinite discontinuity at $x = 0$.

$\therefore \int_{-1}^{3} (1/x^2)\,dx$ doesnot exist.

4) Find the derivative of $y = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\,dt$

Soln:

Let $u = \tan x$. , Then $\frac{d}{dx}\left\{\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt\right\} = \frac{d}{dx}\left\{\int_0^{u} \sqrt{t+\sqrt{t}}\,dt\right\}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{d}{du}\left\{\int_0^{u} \sqrt{t+\sqrt{t}}\,dt\right\}\frac{du}{dx} = \sqrt{u+\sqrt{u}}\sec^2 x$$

Since $\frac{du}{dx} = \sec^2 x$.

$$= \sqrt{\tan x + \sqrt{\tan x}}\,\sec^2 x$$

Indefinite Integral:

In calculus an indefinite integral of a function $f(x)$ is a differentiable function F whose derivative is equal to the original function $f(x)$, i.e, $F' = f(x)$.

Formulae:

1. $\int c\,f(x)\,dx = c\int f(x)\,dx$, c is constant
2. $\int k\,dx = kx + c$.
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$.
4) $\int 1/x\,dx = \log|x| + x$

5. $\int e^x \, dx = e^x + c$

6. $\int a^x \, dx = a^x \dfrac{1}{\log a} + c$

7. $\int \sin x \, dx = -\cos x + c$

8. $\int \cos x \, dx = \sin x + c$

9. $\int \sec^2 x \, dx = \tan x + c$

10. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

11. $\int \sec x \tan x \, dx = \sec x + c$

12. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

13. $\int \dfrac{dx}{x^2+1} = \tan^{-1} x$ (or) $-\cot^{-1}(x) + c$.

14. $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c$ (or) $-\cos^{-1}(x) + c$

15. $\int \sinh x \, dx = \cosh x + c$

16. $\int \cosh x \, dx = \sinh x + c$

17. $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

18. $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

19. $\int \operatorname{sech} x \tan x \, dx = \operatorname{sech} x + c$

20. $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c$

21. $\int \dfrac{dx}{x(x^2-1)} = \sec^{-1}(x) + c$ (or) $\operatorname{cosec}^{-1}(x) + c$.

22. $\int \dfrac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + c$

23. $\int \dfrac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$

24. $\int \dfrac{dx}{x^2-1} = \tanh^{-1} x + c$ (or) $\cosh^{-1}(x) + c$.

1. Evaluate $\int (10x^4 - 2\sec^2 x)\, dx$

Soln:

Let $I = \int (10x^4 - 2\sec^2 x)\, dx$

$= 10\frac{x^5}{5} - 2\tan x + c$

$\therefore I = 2x^5 - 2\tan x + c.$

2. $\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = ?$

Soln:

Let $I = \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

$I = \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\, d\theta$

$= \int \cot\theta \operatorname{cosec}\theta.$

$= -\operatorname{cosec}\theta + c.$

3. Evaluate $\int [\sqrt{x^3} + \sqrt[3]{x^2}]\, dx$

Soln:

Let $I = \int [\sqrt{x^3} + \sqrt[3]{x^2}]\, dx$

Then $I = \int (x^{3/2} + x^{2/3})\, dx$

$I = \frac{x^{5/2}}{5/2} + \frac{x^{5/3}}{5/3} + c$

$= \frac{1}{5}[2x^{5/2} + 3x^{5/3}] + c.$

4. Evaluate $\int [(x^2+1) + \frac{1}{x^2+1}]\, dx$.

Soln:

$\int [x^2+1 + \frac{1}{x^2+1}]\, dx = \frac{x^3}{3} + x + \tan^{-1}x + c.$

5. Evaluate $\int_0^1 (5x - 5^x)\,dx$

Soln:

$$I = \int_0^1 (5x - 5^x)\,dx$$

$$I = 5\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{5^x}{\log 5}\right]_0^1$$

$$= \frac{5}{2}[1-0] - \frac{1}{\log 5}[5^1 - 5^0]$$

$$I = \frac{5}{2} - \frac{4}{\log 5}$$

Problems:

1. Find $\int \frac{dx}{\sin^2 x \cos^2 x}$

Soln:

$$\int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx$$

$$= \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx$$

$$= \tan x + (-\cot x) + c$$

2) $\int \sqrt{1+\sin 2x}\,dx$

Soln:

$$\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}\,dx$$

$$= \int \sqrt{(\cos x + \sin x)^2}\,dx$$

$$= \int (\cos x + \sin x)\,dx$$

$$= \sin x - \cos x + c$$

3. $\int \frac{1}{1-\cos x}\,dx$

Soln:

$$\int \frac{1}{1-\cos x}\,dx = \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x}\,dx.$$

$$= \int \frac{1+\cos x}{1 - \cos^2 x}\, dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\, dx$$

$$= \int (\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x)\, dx$$

$$= -\cot x - \operatorname{cosec} x + c.$$

Properties of Definite Integrals:

We assume that f and g are continuous functions.

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

3. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

6. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ iff $f(2a-x) = f(x)$

7. $\int_0^{2a} f(x)\,dx = 0$ iff $f(2a-x) = -f(x)$.

8. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

9. If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

Problems:

1. Evaluate $\int_{-1}^{1} (2-|x|)\,dx$

Soln:

$$\int_{-1}^{1} (2-|x|)\,dx = \int_{-1}^{1} 2\,dx + \int_{-1}^{1} |x|\,dx$$

Scanned with OKEN Scanner

$= (2x)_{-1}^{1} - 2\int_0^1 x\,dx$ $[\because |x| \text{ is even}]$.

$= 4 - 2\left[\frac{x^2}{2}\right]_0^1$

$= 4 - 1$.

$= 3$.

2) $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$

Soln:

Let $I = \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$

$I = \int_0^{\pi/2} \frac{1}{1+\frac{\sin x}{\cos x}}\,dx$

$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$ —— ①

$= \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx$ —— (2)

① + ② ⇒ $2I = \int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}\,dx$

$= \int_0^{\pi/2} 1\cdot dx = [x]_0^{\pi/2} = \pi/2$

$2I = \pi/2$

$I = \pi/4$.

3) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$

Soln:

Let $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$ —— ①

Then we have, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore I = \int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}}\,dx$$

$$I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\,dx \quad \text{——— (2)}$$

$$(1)+(2) \Rightarrow 2I = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\,dx$$

$$\Rightarrow 2I = \int_0^{\pi} dx \Rightarrow 2I = [x]_0^{\pi} = \pi$$

$$\Rightarrow I = \pi/2.$$

4) Evaluate $\int_{-2}^{2} |x+1|\,dx$

Soln:

Since $|x+1| = \begin{cases} -(x+1), & -2 < x < -1 \\ (x+1), & -1 < x < 2 \end{cases}$

$$\therefore I = \int_{-1}^{1} |x+1|\,dx = -\int_{-2}^{-1} (x+1)\,dx + \int_{-1}^{2} (x+1)\,dx$$

$$\therefore I = -\left[\frac{x^2}{2} + x\right]_{-2}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{2}$$

$$= -\left[\frac{-1}{2} - 0\right] + \left[4 - (-1/2)\right]$$

$I = 5$

5) Evaluate $\int_{-\pi/2}^{\pi/2} \sin^{199} x\,dx$

Soln:

Let $I = \int_{-\pi/2}^{\pi/2} \sin^{199} x\,dx$

Let $f(x) = \sin^{199} x$

Then $f(-x) = \sin^{199}(-x) = -\sin^{199} x = -f(x)$

$\therefore f(x)$ is an odd function.

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^{199} x \, dx = 0.$$

Integration by substitution rule:

Type : I - (A)

$$\int [f(x)]^n f'(x)\,dx \text{ (or) } \int \phi(x) f'(x)\,dx \quad \text{——} (1)$$

Substitute $u = f(x)$,

$du = f'(x)\,dx$

(1) $\Rightarrow \int u^n du$ (or) $\int \phi(u)\,du$ and then proceed.

1. Solve $\int \frac{1}{x^2}\sqrt{2-1/x}\,dx$

Soln:

Let $I = \int \frac{1}{x^2}\sqrt{2-1/x}\,dx$

Let $u = 2 - 1/x$ $\quad du = \frac{1}{x^2}dx$

$\therefore I = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C.$

Hence $I = 2/3\,(2-1/x)^{3/2} + C.$

2) Evaluate $\int \frac{1}{(3x-4)^{3/2}}\,dx$

Soln:

Let $I = \int \frac{1}{(3x-4)^{3/2}}\,dx$

Let $u = 3x-4$, $du = 3dx$

$\Rightarrow \frac{du}{3} = dx$

$\therefore I = \int \frac{du/3}{u^{3/2}} = \frac{-2}{3}\frac{1}{\sqrt{u}} + C$

$\Rightarrow I = \frac{-2}{3}\frac{1}{\sqrt{3x-4}} + C.$

Type : I : B [logarithmic function]

1. Evaluate $\int_1^e \frac{\log x}{x}\, dx$

Soln:

Let $I = \int_1^e \frac{\log x}{x}\, dx$

x	1	e
u	0	1

Let $u = \log x \quad du = \frac{dx}{x}$.

$\therefore I = \int_0^1 u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$

2. Evaluate $\int \frac{\sec^2(\log x)}{x}\, dx$

Soln:

Let $I = \int \frac{\sec^2(\log x)}{x}\, dx$

Let $u = \log x \quad du = \frac{dx}{x}$

$\therefore I = \int \sec^2 u\, du = \tan u + c$.

$\therefore I = \tan(\log x) + c$.

Type:I (c). [Exponential function]

1. Evaluate $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

Soln:

Let $I = \int_1^2 \frac{e^{1/x}}{x^2}\, dx$

x	1	2
u	e	$e^{1/2}$

Let $u = e^{1/x}$; $du = e^{1/x}(-1/x^2)\, dx$

$\Rightarrow -du = \frac{e^{1/x}}{x^2}\, dx$

$\therefore I = \int_e^{e^{1/2}} -du = -[u]_e^{e^{1/2}} = e - \sqrt{e}$

2) Evaluate $\int e^{\cos x} \sin x \, dx$

Soln:

Let $I = \int e^{\cos x} \sin x \, dx$

$u = e^{\cos x}$ $\quad du = e^{\cos x}(-\sin x)dx$

$\therefore I = \int (-du) = -u + c = e^{-\cos x} + c$.

3) Evaluate $\int (\log a)^x \, dx$

Soln:

Let $I = \int (\log a)^x \, dx$.

Then, $I = \int e^{\log [\log a]^x} dx$

$= \int e^{x \log(\log a)} dx$.

$= \dfrac{e^{x \log(\log a)}}{\log(\log a)} + c$

$I = \dfrac{(\log a)^x}{\log(\log a)} + c$

Type: 1(D) (Trigonometric functions).

1. Evaluate $\int_0^{\pi/2} \cos x \sin(\sin x) dx$

Soln:

Let $I = \int_0^{\pi/2} \cos x \, (\sin(\sin x)) \, dx$

x	0	$\pi/2$
u	0	1

Put $u = \sin x$, $\quad du = \cos x \, dx$

$\therefore I = \int_0^1 \sin u \, du$

$= [-\cos u]_0^1$

$= -\cos 1 + \cos 0$

$= 1 - \cos 1$

· Techniques of Integration:

Integration by parts : Method:

$$\int u\,dv = uv - \int v\,du$$

1. Solve $\int_0^1 \frac{y}{e^{2y}}\,dy$

Soln:

Let $I = \int_0^1 \frac{y}{e^{2y}}\,dy = \int_0^1 y e^{-2y}\,dy$

Let $u = y$, $dv = e^{-2y}dy$

$du = dy$ $\quad v = e^{-2y}/-2$

$$\therefore I = \left[y \frac{e^{-2y}}{-2}\right]_0^1 + \int_0^1 \frac{e^{-2y}}{2}\,dy$$

$$= \frac{-1}{2}e^{-2} + \frac{1}{2}\left[\frac{e^{-2y}}{-2}\right]_0^1$$

$$= \frac{-1}{2}e^{-2} + \frac{1}{2}\left[\frac{-e^{-2}}{2} + \frac{1}{2}\right]$$

$$\Rightarrow \frac{1}{4} - \frac{3}{4}e^{-2}$$

Problems based on trigonometric integrals:

1. Evaluate $\int \frac{x}{1+\sin x}\,dx$

Soln:

Let $I = \int \frac{x}{1+\sin x}\,dx$

Then $I = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$

$$= \int \frac{(x - x\sin x)}{\cos^2 x}\,dx \quad \left[\because 1-\sin^2 x = \cos^2 x\right]$$

$$I = \int (x\sec^2 x - x\sec x\tan x)\, dx \quad \text{—(1)}$$

Take $\int x\sec^2 x\, dx = x\tan x - \int \tan x\, dx$

$u = x$
$du = dx$
$dv = \sec^2 x\, dx$
$v = \int \sec^2 x\, dx$

$$= x\tan x - \log(\sec x)$$

Take $\int x\sec x\tan x\, dx$

$u = x, \quad dv = \sec x\tan x\, dx$

$du = dx \qquad v = \sec x$

$\therefore \int x\sec x\tan x\, dx = x\sec x - \log(\sec x + \tan x)$

$$I = \int \frac{x}{1+\sin x}\, dx$$

$$= x\tan x - \log(\sec x) - x\sec x + \log(\sec x + \tan x) + C.$$

2) Evaluate $\int_0^1 \tan^{-1} x\, dx$

Soln:

Let $I = \int_0^1 \tan^{-1} x\, dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{dx}{1+x^2}, \quad v = x.$

$$\therefore I = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2}$$

$$= \frac{\pi}{4} - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Scanned with OKEN Scanner

CHAPTER - 5

MULTIPLE INTEGRALS

Double integrals in cartesian coordinates:

Type : I (limits are constants).

1. Evaluate $\int_0^a \int_0^b xy\,(x-y)\,dx\,dy$

Soln:

$$\int_0^a \int_0^b xy\,(x-y)\,dx\,dy = \int_0^a \int_0^b (x^2y - xy^2)\,dx\,dy$$

$$= \int_0^a \left[\frac{x^3y}{3} - \frac{x^2y^2}{2} \right]_0^b dy$$

$$= \int_0^a \left[\frac{b^3y}{3} - \frac{b^2y^2}{2} \right] dy$$

$$= \left[\frac{b^3y^2}{6} - \frac{b^2y^3}{6} \right]_0^a$$

$$= \left\{ \frac{b^2y^2}{6} \left[b - y \right] \right\}_0^a$$

$$= \frac{b^2a^2}{6}(b-a)$$

2. Evaluate $\int_2^a \int_2^b \frac{dx\,dy}{xy}$

Soln:

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \int_2^a \frac{dx}{x} \int_2^b \frac{dy}{y}$$

$$= \left[\log x \right]_2^a \left[\log y \right]_2^b$$

$$= (\log a - \log 2)\,(\log b - \log 2)$$

$$= \log(a/2) \cdot \log(b/2).$$

Type: 2 (Limits are variables)

1) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$.

Soln:

Put $\sqrt{1+x^2} = t$

Dont find 'dt' because $t = \sqrt{1+x^2}$ is constant w.r.to y.

$$I = \int_0^1 \int_0^t \frac{dy\,dx}{t^2+y^2} = \int_0^1 \left[\frac{1}{t}\tan^{-1} y/t\right]_0^t dx$$

$$= \int_0^1 \frac{1}{t}\left(\tan^{-1} 1 - \tan^{-1} 0\right) dx = \int_0^1 \frac{1}{t}\frac{\pi}{4}\,dx \qquad \left[\because t = \sqrt{1+x^2}\right]$$

$$= \frac{\pi}{4} \cancel{\log} \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \frac{\pi}{4}\log\left[x+\sqrt{1+x^2}\right]_0^1 = \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log 1\right]$$

$$I = \frac{\pi}{4}\log(1+\sqrt{2}). \qquad [\because \log 1 = 0]$$

H.W 1. Evaluate $\int_1^2 \int_0^{x^2} x\,dy\,dx$. Ans $15/4$

2. Evaluate $\int_0^1 \int_0^2 xy(x+y)\,dx\,dy$. Ans $I = 2$.

2. Evaluate $\int_0^1 \int_y^{y^2+1} x^2 y\,dx\,dy = \int_0^1 y\left(\frac{x^3}{3}\right)_y^{y^2+1} dy$

$$= \frac{1}{3}\int_0^1 y\left\{(y^2+1)^3 - y^3\right\} dy$$

$$= \frac{1}{3}\int_0^1 y\left\{y^6+1+3y^4+3y^2-y^3\right\} dy$$

$$= \frac{1}{3}\left\{\frac{y^8}{8}+\frac{y^2}{2}+\frac{3y^6}{6}+\frac{3y^6}{6}+\frac{3y^4}{4}-\frac{y^5}{5}\right\}_0^1$$

$$= \frac{1}{3}\left\{\frac{1}{8}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}-\frac{1}{5}-0\right\}$$

$$= \frac{1}{3}\left\{\frac{1}{8}+1+\frac{3}{4}-\frac{1}{5}\right\}$$

$$= \frac{1}{3}\left(\frac{67}{40}\right)$$

$$= \frac{67}{120}$$

Sketch the region of integration:

1. Sketch the region of integration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx\, dy$

Soln:

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx\, dy = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dy\, dx \quad \text{[correct form]}$$

$$y = \sqrt{ax-x^2} \rightarrow y = \sqrt{a^2-x^2}$$

$$y^2 = ax - x^2 \rightarrow y^2 = a^2 - x^2$$

$$x^2+y^2-ax=0 \rightarrow y^2+x^2=a^2$$

$$x^2+y^2-ax+\frac{a^2}{4}-\frac{a^2}{4}=0 \rightarrow x^2+y^2=a^2$$

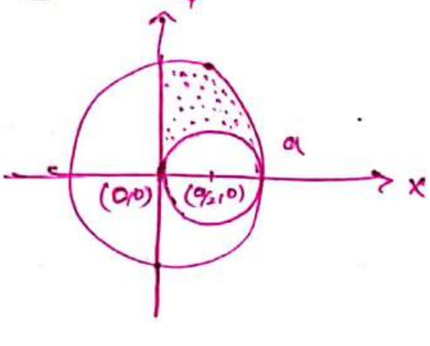
$$\left(x-\frac{a}{2}\right)^2+(y-0)^2=\left(\frac{a}{2}\right)^2 \rightarrow x^2+y^2=a^2$$

circle, $C = (a/2, 0)$ & $\rightarrow$ circle, $C=(0,0)$ & radi $= a$

radi $= a/2$

Also, $x : 0 \rightarrow a$

$x = 0 \rightarrow x = a$.



2) Sketch the region $\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\, dy$

Soln:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\, dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx \quad \text{(correct form)}$$

Given $y = 0 \rightarrow \sqrt{a^2-x^2}$

$$y = 0 \rightarrow y = \sqrt{a^2-x^2}$$

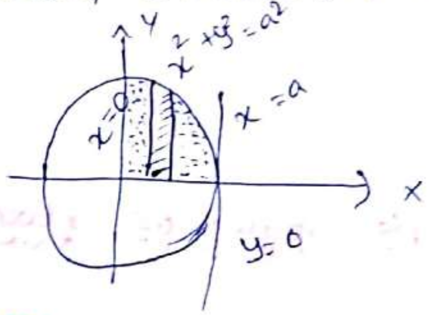
$$y^2 = a^2 - x^2$$

$y^2 = a^2 - x^2$

$x^2+y^2 = a^2$ circle, $c = (0,0)$ & rad $= a$.

Also $x : 0 \to a$

$x = 0 \to x = a$



Change of order of integration:

When we change the order of integration the limits are also changed. The following points are very useful when the change of order of integration.

1. If the limits of the inner integral is a function of x. (or function of y) then the first integration should be w.r.to y (or w.r.t x).

2. Draw the region of integration by using the given limits.

3. If the integration is first w.r.to x keeping 'y' as a constant Then consider the horizontal strip and vice versa.

4) Then change into vertical strip, to find the limits of x and y

Problems: 1.

1) change the order of integration for $\int_0^1 \int_0^2 f(x,y)\, dx\, dy$.

Soln:

Since the limits of the given integral are constant, change the order is nothing but the interchanging the limits for the corresponding variables.

$\therefore$ Changing the order is, $I = \int_0^2 \int_0^1 f(x,y)\, dy\, dx$.

2) change the order of integration in $I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and evaluate.

Soln:

Step:1 To plot the region.

The region of integration is bounded by $x=y$, $x=a$, $y=0$, $y=a$.

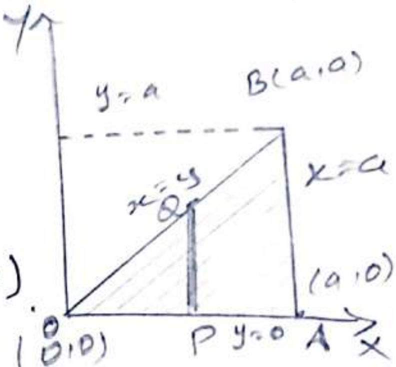
$x=y$, a line passing through the origin intersect xoy plane.

$x=a$, a line $\perp^r$ to x-axis.

$y=0$ is x-axis

$y=a$, a line perpendicular to y-axis.

The region of integration is OABO (shaded region)



Step:2 To evaluate the integral:

After changing the order, first integrate w.r.to y, then x.

Since the order of integration is dydx, draw a vertical strip PQ.

At P, $y=0$ At Q, $y=x$

At O, $x=0$ At A, $x=a$.

$$I = \int_0^a \int_0^x \frac{x}{x^2+y^2}\,dy\,dx = \int_0^a \int_0^x \frac{dy}{y^2+x^2}\cdot x\,dx. \quad \left[\begin{array}{l}\tan^{-1}1 = \pi/4 \\ \tan^{-1}0 = 0\end{array}\right]$$

$$= \int_0^a \left[\frac{1}{\cancel{x}}\tan^{-1}(y/x)\right]_0^x \cancel{x}\,dx = \int_0^a [\tan^{-1}1 - \tan^{-1}0]\,dx$$

$$= \int_0^a \pi/4\,dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4} - 0$$

$$I = \frac{\pi a}{4}$$

2) Evaluate by changing the order of integration, $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$.

Soln:

Step:1 To plot the region.

The region of integration is bounded by $y=0$, $y=3$

$x=1$, $x=\sqrt{4-y} \Rightarrow x^2 = 4-y \Rightarrow x^2 = -(y-4)$.

$y=0$ is x-axis; $y=3$ is a straight line perpendicular to y-axis.

$x^2 = -(y-4)$ is a parabola symmetrical about y-axis with vertex as at $(0,4)$.

The region of integration is ABC (shaded region)

$y = 4-x^2$		
x	1	2
y	3	0

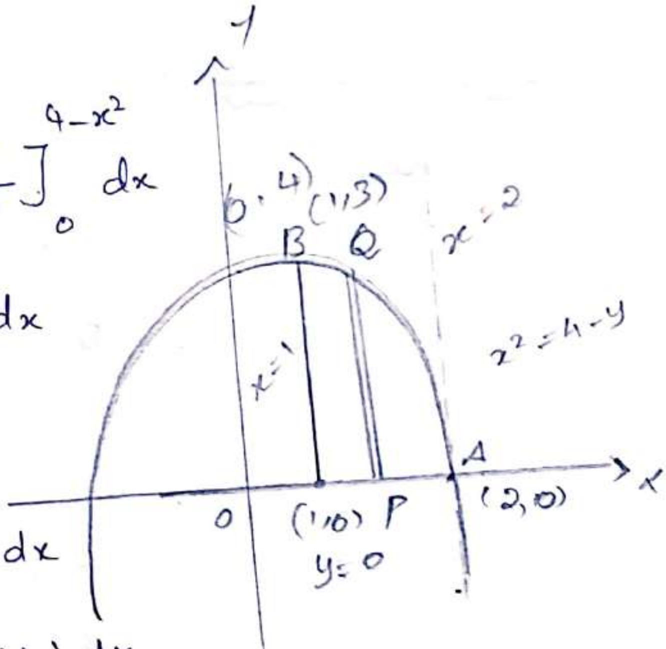
Step:2 To evaluate the integral

After changing the order, first integrate w.r.to y then x.

Since the order of integration is dydx, draw vertical strip PQ.

At P, $y=0$ At Q, $y = 4-x^2$

At C, $x=1$ At A, $x=2$



$$I = \int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx = \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_1^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2} \right) dx$$

$$= \int_1^2 \left(4x - x^3 + \frac{16 - 8x^2 + x^4}{2} \right) dx$$

$$= \int_1^2 \left(\frac{8x - 2x^3 + 16 - 8x^2 + x^4}{2} \right) dx$$

$$= \frac{1}{2} \int_1^2 (x^4 - 2x^3 - 8x^2 + 8x + 16)\,dx$$

$$= \frac{1}{2} \left[\frac{x^5}{5} - \frac{2x^4}{4} - \frac{8x^3}{3} + \frac{8x^2}{2} + 16x \right]_1^2$$

$$= \frac{1}{2} \left(\frac{32}{5} - 8 - \frac{64}{3} + 16 + 32 - \frac{1}{5} + \frac{1}{2} + \frac{8}{3} - 4 - 16 \right)$$

$$= \frac{1}{2} \left(\frac{31}{5} - \frac{56}{3} + 20 + \frac{1}{2} \right)$$

$$= \frac{241}{60}$$

H.W Change the order of integration in $\int_0^b \int_0^{a/b(b-y)} xy\,dx\,dy$ and hence evaluate it.

Ans $I = \frac{2}{3} a^4$

4) Change the order of integration for $\int_0^a \int_0^x f(x,y)\,dy\,dx$

Soln:

Step:1 To plot the region.

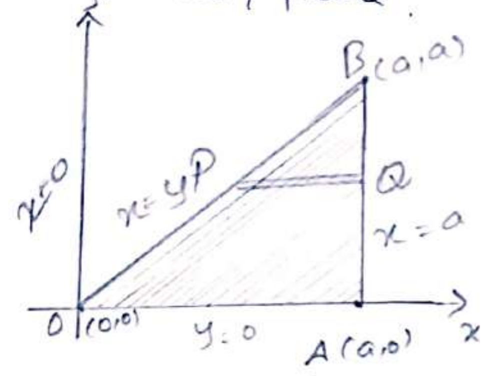
$y=0$ is x axis

$y=x$, passing through the origin and bisect the XOY plane

$x=0$ is y-axis;

$x=a$, a line $\perp r$ to x axis.

The region of integration is OABO.



Step:2 To change the order.

After changing the order, first integrate w.r.to x and y.

Since the order of integration is $dx\,dy$, draw a horizontal strip PQ.

At P, $x=y$, At Q, $x=a$

At O, $y=0$, At B, $y=a$.

Change of order is $I = \int_0^a \int_y^a f(x,y)\,dx\,dy$.

6. By changing the order of integration, evaluate $\int_0^a \int_x^a (x^2+y^2)\,dy\,dx$.

Soln:

Step:1 To plot the region.

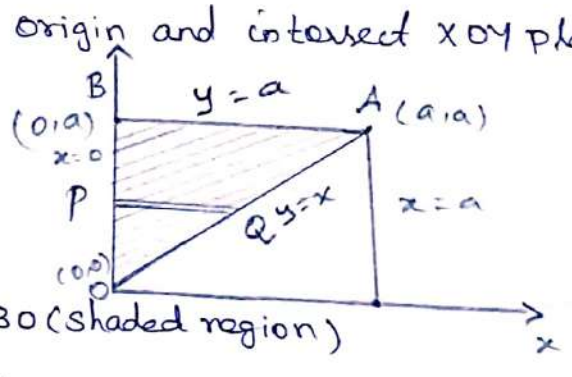
The region of integration is bounded by $y=x$, $y=a$, $x=0$, $x=a$.

$y=x$, a line passing through the origin and intersect XOY plane

$y=a$, a line $\perp r$ to y-axis.

$x=0$, is y-axis

$x=a$, a line $\perp r$ to x-axis.



The region of integration is OABO (shaded region)

Step:2 To evaluate the integral:

After changing the order first integrate w.r.to x then y.

Since the order of integration is $dx\,dy$, draw a horizontal strip PQ

At P, $x=0$; At Q, $x=y$

At O, $y=0$ At B, $y=a$

$$I = \int_0^a \int_0^y (x^2+y^2)\,dx\,dy = \int_0^a \left[\frac{x^3}{3} + xy^2\right]_0^y dy$$

$$= \int_0^a \left[\frac{y^3}{3} + y^3\right] dy = \frac{4}{3}\int_0^a y^3\,dy$$

$$= \frac{4}{3}\left[\frac{y^4}{4}\right]_0^a$$

$$I = \frac{a^4}{3}.$$

7. By changing the order of integration evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$.

Soln:

Step:1 To plot the region:

The region of integration is bounded by $x=0$, $x=4a$.

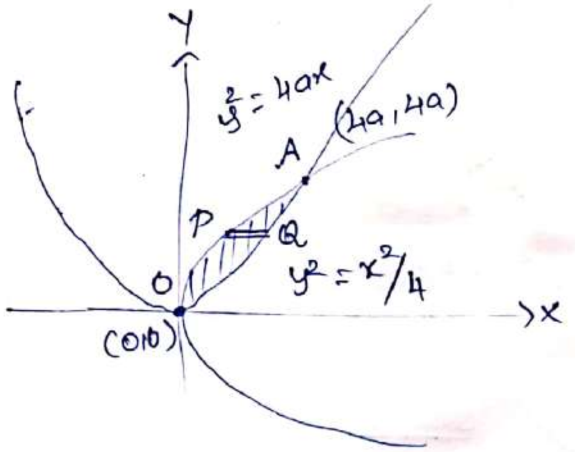
$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$ ——(1)

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ ——(2)

$x=0$ is y-axis; $x=4a$ is a straight line $\perp$r to y-axis

$x^2 = 4ay$ is a parabola symmetrical about y-axis with vertex at the origin.

$y^2 = 4ax$ is a parabola symmetrical about x-axis with vertex at the origin.



To find the point of intersection:

From (2), $x = \frac{y^2}{4a}$ ——(3).

substitute (3) in (1)

$\left(\frac{y^2}{4a}\right)^2 = 4ay$

$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 = 64a^3y$.

$\therefore \frac{y^4}{16a^2} = 4a \quad y^4 - 64a^3y = 0$

$y(y^3 - 64a^3) = 0$

$y = 0, \; y^3 - 64a^3 = 0$

$y^3 = 64a^3 \Rightarrow y = 4a$

$y = x^2/4a$		
x	0	$4a$
y	0	$4a$

$\therefore y = 0, y = 4a.$

$\therefore$ The points of intersection are $(0,0)(4a,4a)$.

step: 2 To evaluate the integral.

After changing the order, first integrate w.r.to x, then y. Since the order of integration is $dx\,dy$, draw a horizontal strip PQ:

At P, $x = \frac{y^2}{4a}$; At Q $x = 2\sqrt{a}\sqrt{y}$

At O, $y = 0$ At A, $y = 4a$.

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{a}\sqrt{y}} dx\,dy = \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{a}\sqrt{y}} dy$$

$$= \int_0^{4a} \left(2\sqrt{a}\sqrt{y} - \frac{y^2}{4a}\right) dy$$

$$= \int_0^{4a} \left(2\sqrt{a}\, y^{1/2} - \frac{y^2}{4a}\right) dy$$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{4}{3}\sqrt{a}\; 4a\sqrt{4a} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$I = \frac{16a^2}{3}.$$

H.W changing the order of integration $\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} x^2\, dy\, dx$ and then integrate it..

Ans $I = \int_0^b \int_0^{a/b\sqrt{b^2-y^2}} x^2\, dx\, dy = \frac{a^3 b\pi}{16}$.

Problems with two sub-regions after changing the order of integration

1. change the order of integration $I = \int_0^1 \int_{x^2}^{2-x} dx\, dy$ and hence evaluate it.

Soln:

Since inner limit is a function of x, first integrate w.r.to y then x.

$$\therefore I = \int_0^1 \int_{x^2}^{2-x} dy\, dx$$

Step:1 To plot the region:

The region of integration is bounded by

$y = x^2$, $y = 2-x \Rightarrow x+y=2$, $x=0$, $x=1$

$y = x^2$ is a parabola symmetrical about y axis.

$x+y=2$ is a straight line intersect the coordinate axes at $(2,0)$ and $(0,2)$.

$x=0$ is y-axis $x=1$, a line $\perp$ to x-axis.

The region of integration is OABCO (Shaded region).

Step 2 To evaluate the integral,

After changing the order, first integrate w.r.to x then y.

Since there are two slopes for horizontal strip, the region is splited into two parts R_1 and R_2, draw horizontal strips P_1Q_1, P_2Q_2.

In R_1

At P_1, $x=0$, At Q_1, $x=\sqrt{y}$

At O, $y=0$, At A, $y=1$

In R_2

At P_2, $x=0$, At $Q_2 = x = 2-y$

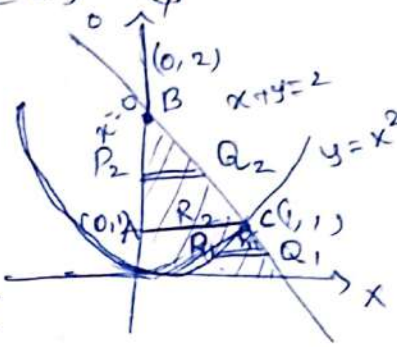
At A, $y=1$, At B, $y=2$.

Scanned with OKEN Scanner

$\therefore I = \int_0^1 \int_0^{\sqrt{y}} dx\,dy + \int_1^2 \int_0^{2-y} dx\,dy = \int_0^1 [x]_0^{\sqrt{y}}\,dy + \int_1^2 [x]_0^{2-y}\,dy$

$= \int_0^1 \sqrt{y}\,dy + \int_1^2 (2-y)\,dy$

$= \left[\frac{y^{3/2}}{3/2}\right]_0^1 + \left[2y - \frac{y^2}{2}\right]_1^2 = \frac{2}{3} + 4 - 2 - 2 + \frac{1}{2}$



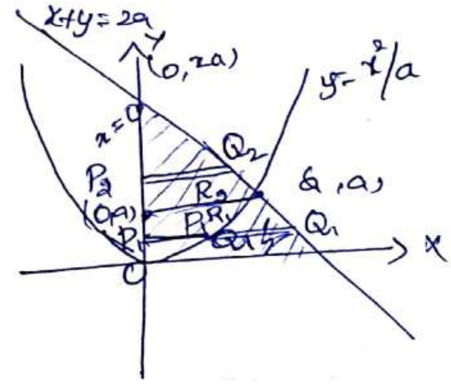
$= 7/6$

2. Change the order of integration $I = \int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$ and hence evaluate it.

Soln:

Since inner limit is a function of x, first integrate w.r.to y then x.

$\therefore I = \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$.



Step: 1 To plot the region:

The region of integration is bounded by

$y = \frac{x^2}{a}$, $y = 2a - x \Rightarrow x + y = 2a$, $x = 0$, $x = a$.

$y = x^2$ is a parabola symmetrical about y-axis. $x+y = 2a$ is a straight line intersect the coordinate axes at $(2a, 0)$ and $(0, 2a)$

$x = 0$ is y-axis

$x = a$, a line perpendicular to x-axis

Step: 2 To evaluate the integral:

After changing the order, first integrate w.r.to 'x' then 'y'.

Since there are two slopes for horizontal strip, the region is spitted into two parts R_1 & R_2.

Draw a horizontal strips P_1Q_1, P_2Q_2

In R_1

At P_1, $x=0$, At Q_1, $x=\sqrt{ay}$

At O, $y=0$; At A, $y=a$

In R_2.

At P_2, $x=0$; At Q_2, $x=2a-y$

At A, $y=a$, At B, $y=2a$.

$$I=\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx=\int_0^a\int_0^{\sqrt{ay}} xy.dy\,dx+\int_0^{2a}\int_0^{2a-y} xy\,dy\,dx$$

$$=\int_0^a y\left[x^2/2\right]_0^{\sqrt{ay}} dy+\int_0^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy=\frac{1}{2}\int_0^a ay^2\,dy+\frac{1}{2}\int_a^{2a} y(2a-y)^2 dy$$

$$=\frac{1}{2}a\int_0^a y^2\,dy+\frac{1}{2}\int_0^{2a} y(4a^2-4ay+y^2)\,dy$$

$$=\frac{a}{2}\left[\frac{y^3}{3}\right]_0^a+\frac{1}{2}\int_0^{2a} 4a^2y-4ay^2+y^3\,dy$$

$$=\frac{a^4}{6}+\frac{1}{2}\left[4a^2\frac{y^2}{2}-4a\frac{y^3}{3}+\frac{y^4}{4}\right]$$

$$=\frac{a^4}{6}+\frac{1}{2}\left[8a^4-\frac{32a^4}{3}+\frac{16a^4}{4}-2a^4+\frac{4a^4}{3}-\frac{a^4}{4}\right].$$

$$=\frac{a^4}{6}+\frac{1}{2}\left[\frac{72a^4+45a^4-112a^4}{12}\right]$$

$$=\frac{a^4}{6}+\frac{1}{24}(5a^4)=\frac{9a^4}{24}$$

$$I=\frac{3a^4}{8}$$

Double Integrals in polar Coordinates

Problems:

1) calculate $\iint r^2\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$.

Soln:

Given $r=2\sin\theta$ —(1) is a circle of with diameter 2 passing through the origin symmetric about $\theta=\pi/2$

$r = 4\sin\theta$ —— (2) is a circle with diameter 4 passing through (7)
the origin symmetric about $\theta = \pi/2$.
The shaded area between these circles is the region of integrati

Area = 2 × Area of region in the first quadrant.

$$= 2 \iint_{OABO} r\,dr\,d\theta$$

At P, $r = 2\sin\theta$; At Q, $r = 4\sin\theta$

θ varies from 0 to $\pi/2$ [Region is in first quadrant]

$$\text{Area} = 2\int_0^{\pi/2}\int_{2\sin\theta}^{4\sin\theta} r^2\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2}(64\sin^3\theta - 8\sin^3\theta)\,d\theta$$

$$= \frac{112}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{112}{3}\cdot\frac{2}{3}$$

$$\left[\because \int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{2}{3}\cdot 1 = \frac{2}{3}\right]$$

$$\text{Area} = \frac{224}{9}$$

Changing cartesian into polar system:

$x = r\cos\theta$

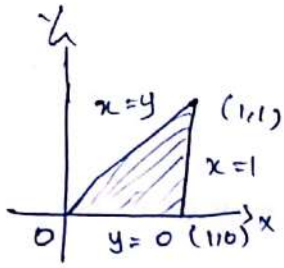
$y = r\sin\theta$

$x^2 + y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$.

1. Change into polar coordinates $\int_0^1\int_0^x f(x,y)\,dy\,dx$.

Soln:



The polar form is, $x = r\cos\theta$, $y = r\sin\theta$.

$x^2 + y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

Inner upper limit is

$y = x \Rightarrow r\sin\theta = r\cos\theta \Rightarrow \sin\theta = \cos\theta$

Scanned with OKEN Scanner

$\tan\theta = 1$; $\theta = \pi/4$

$\therefore$ θ varies from $\theta = 0$ to $\theta = \pi/4$

Outer upper limit $x = 1 \Rightarrow r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta}$

r varies from $r = 0$ to $r = \frac{1}{\cos\theta}$

$$I = \int_0^{\pi/4}\int_0^{1/\cos\theta} f(r,\theta)\, r\,dr\,d\theta$$

2) Changing into polar coordinates $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

<u>Soln:</u>

The polar form is $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$.

Inner upper limit is $y = \sqrt{a^2-x^2}$

ie, $x^2+y^2 = a^2 \Rightarrow r^2 = a \Rightarrow r = a$

Since both the lower limits are negative the region is entire circle $x^2+y^2 = a^2$

$\therefore$ θ varies from $\theta = 0$ to 2π

$$I = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta.$$

3) Evaluate by changing to polar coordinates $\int_0^{a}\int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

<u>Soln:</u>

<u>To plot the region:</u> ~~is bo~~

The region is bounded by $x = 0, x = a, y = 0, y = \sqrt{a^2-x^2}$

ie, $x^2+y^2 = a^2$

$x = 0$ is y-axis, $x = a$ is a line $\perp^r$ to x-axis.

$y = 0$ is x-axis, $y = a$ is a line $\perp^r$ to y-axis.

The region of integration is OABO (Shaded region)

Step: 2 To evaluate the integration:

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$x^2 + y^2 = r^2$.

Since both the lower limits are 0, the region lies in the first quadrant.

θ varies from $\theta = 0$ to $\theta = \pi/2$.

$x^2+y^2 = a^2 \Rightarrow r^2 = a \Rightarrow r = a$

$\therefore$ r varies from $r = 0$ to $r = a$.

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^a \sqrt{r^2} \cdot r \cdot dr\, d\theta$$

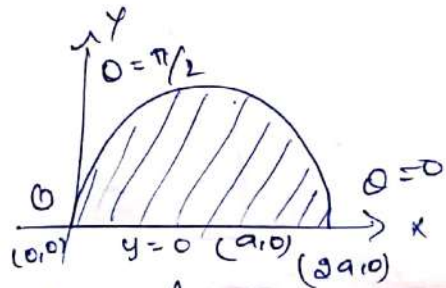
$$= \int_0^{\pi/2} \int_0^a r^2\, dr\, d\theta = \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^a d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} d\theta = \frac{a^3}{3} [\theta]_0^{\pi/2}$$

$$I = \frac{\pi a^3}{6}$$

Evaluation of double integrals using polar coordinates:

1) Evaluate $\iint (x+y)\,dx\,dy$ over the region in the positive quadrant bounded by the circle $x^2 - 2ax + y^2 = 0$



Soln:

Step: 1 To plot the region:

Given $x^2 - 2ax + y^2 = 0$

$\Rightarrow (x-a)^2 + y^2 = a^2$, which is a circle with centre $(a,0)$ and radius 'a'.

The region of integration OAO, Since the region is circle we use polar coordinate.

Step 2 To evaluate the integral:

polar form is $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$; $dxdy = r\,dr\,d\theta$

we have $x^2+y^2-2ax = 0$

$\Rightarrow r^2 - 2ar\cos\theta = 0$

$r(r-2a\cos\theta) = 0$

$r = 0$ (or) $r - 2a\cos\theta = 0$

$r = 0$ (or) $r = 2a\cos\theta$.

$\therefore$ θ varies from $r=0$ to $r = 2a\cos\theta$.

Since the region is a semi-circle lies in the first quadrant.

θ varies from $\theta = 0$ to $\theta = \pi/2$.

$$I = \iint (x+y)\,dx\,dy = \int_0^{\pi/2}\int_0^{a\cos\theta} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} (\cos\theta + \sin\theta)\, r^2\,dr\,d\theta = \int_0^{\pi/2} (\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{a\cos\theta} dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\cos\theta+\sin\theta)\, a^3\cos^3\theta\, d\theta = \frac{a^3}{3}\int_0^{\pi/2} (\cos^4\theta + \cos^3\theta\sin\theta)\,d\theta$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2} \cos^4\theta\, d\theta + \int_0^{\pi/2} \cos^3\theta \sin\theta\, d\theta\right]$$

$$= \frac{a^3}{3}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \int_0^{\pi/2} \cos^3\theta\; d(\cos\theta)\right]$$

$$= \frac{a^3}{3}\left[\frac{3\pi}{16} + \left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2}\right]$$

$$= \frac{a^3}{3}\left[\frac{3\pi}{16} + 0 - \frac{1}{4}\right] \qquad \left[\because \cos\pi/2 = 0,\ \cos 0 = 1\right]$$

$$= \frac{a^3}{3}\left[\frac{3\pi}{16} - \frac{1}{4}\right] = \frac{a^3}{3}\left(\frac{3\pi-4}{16}\right)$$

$$I = \frac{(3\pi-4)a^3}{48}$$

Area as double Integral

The area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx .$$

Similarly, the area enclosed by $x = \phi_1(y)$ and $x = \phi_2(y)$ and $y = c$, $y = d$ is given by,

$$\text{Area} = \int_c^d \int_{\phi_1(x)}^{\phi_2(x)} dx\,dy .$$

Problems:

~~1. Evaluate $\iint (x+y)\,dx\,dy$ over the region in the positive quadrant bounded by the circle $x^2 - 2ax + y^2 = 0$~~

~~Soln:~~

~~Step: 1 To plot the region.~~

~~Given $x^2 - 2ax + y^2 = 0$.~~

1. Find the area of the region enclosed by the curves $y = x$ and $y = x^2$.

Soln:

Step: 1 (To plot the region)

$y = x$ is a straight line passes through the origin and intersect the XOY-plane.

$y = x^2$ is a parabola symmetrical about y-axis with vertex $(0,0)$.

The region of integration is OAO (shaded region)

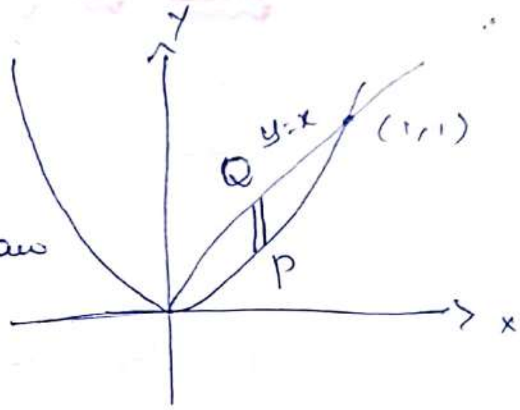
To find the point of intersection:

Given $y = x \longrightarrow (1)$, $y = x^2 \longrightarrow (2)$

$y = x$		
x	0	1
y	0	1

The points of intersections are $(0,0)$ $(1,1)$

<u>Step:2</u> <u>To evaluate the area</u>.



$$\text{Area} = \iint_R dy\,dx$$

Since the order of integration is $dy\,dx$, draw a vertical strip PQ.

At P, $y = x^2$; At Q, $y = x$

At O, $x = 0$, At A, $x = 1$.

$$\text{Area} = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx$$

$$= \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

Area $= \frac{1}{6}$ sq. units.

2) Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

<u>Soln:</u>

<u>Step:1</u> <u>To plot the region:</u>

$y^2 = 4ax$ is a parabola symmetrical about x-axis and passing through the origin.

$x^2 = 4ay$ is a parabola symmetrical about y-axis and passing through the origin.

The region of integration is OAO (shaded region)

<u>To find the points of intersection:</u>

$y^2 = 4ax \Rightarrow x = \frac{y^2}{4a}$ ——— (1)

$x^2 = 4ay$ ——— (2)

(2) $\Rightarrow \left(\frac{y^2}{4a}\right)^2 = 4ay$ (using (1))

$x = y^2/4a$

x	0	$4a$
y	0	$4a$

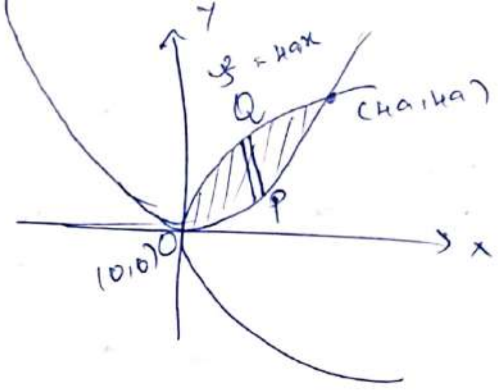
$\therefore \frac{y^4}{16a^2} = 4ay$

$y^4 = 64a^3y$

$y^4 - 64a^3y$

$y(y^3 - 64a^3) = 0$

$y = 0, \quad y^3 = 64a^3 \Rightarrow y = 4a$



The points of intersection are $(0,0), (4a,4a)$

Step:2 To evaluate the area.

$$\text{Area} = \iint_R dy\,dx$$

Since the order of integration is $dy \cdot dx$, draw a vertical strip PQ.

At P, $y = \frac{x^2}{4a}$, At Q $y = 2\sqrt{a}\sqrt{x}$

At O, $x = 0$, At A, $x = 4a$.

$$\text{Area} = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{a}\sqrt{x}} dy\,dx = \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{a}\sqrt{x}}\,dx$$

$$= \int_0^{4a}\left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a}\right)dx = \int_0^{4a}\left(2\sqrt{a}\,x^{1/2} - \frac{x^2}{4a}\right)dx$$

$$= \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a} = 2\sqrt{a}\,\frac{(4a)^{3/2}}{3/2} - \frac{(4a)^3}{12a}$$

$$= \frac{4\sqrt{a}}{3}4a\sqrt{4a} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16}{3}a^2$$

$$\text{Area} = \frac{16a^2}{3} \text{ sq. units.}$$

HW Find the area between the two parabolas $3y^2 = 25x$ and $5x^2 = 9y$.

Ans: The points of intersection: $(0,0), (3,5)$

Area = 5 sq. units.

3. Find the area outside to the circle $r = a$ and inside to the cardioid $r = a(1+\cos\theta)$.

Soln:

Step: 1) To plot the region.

$r = a$, a circle with centre $(0,0)$ and radius a.

$r = a(1+\cos\theta)$ a cardioid, symmetrical about the initial line $\theta = 0$ and passes through the origin $(0,0)$ ~~and symmetrical~~

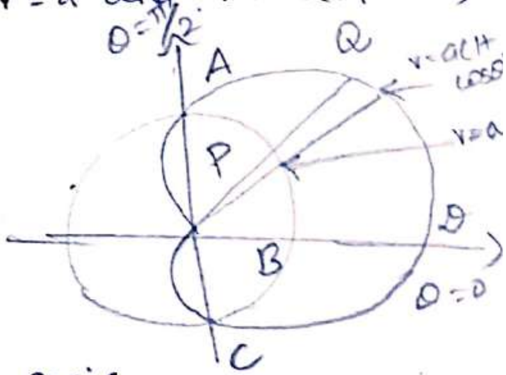
The region of integration is ABCDA (shaded region)

The region is symmetric about the line $\theta = 0$.

Area $= 2 \times$ Area of the region above the initial line.

$$= 2 \iint_{ABDA} r\,dr\,d\theta.$$

The point of intersection of the circle $r = a$ and $r = a(1+\cos\theta)$ are obtained as,



$$a = a(1+\cos\theta)$$

$$\cos\theta = 0$$

$$\Rightarrow \theta = \pm \pi/2.$$

Since the region ABDA lies in the first quadrant and touches the x-axis & y-axis,

θ varies from $\theta = 0$ to $\theta = \pi/2$.

Step: 2 To evaluate the integral.

$$\text{Area} = 2 \iint_{ABDA} r\,dr\,d\theta$$

Limits of r: At P, $r = a$; At Q $r = a(1+\cos\theta)$

Limits of θ : $\theta = 0$; $\theta = \pi/2$.

$$I = 2\int_0^{\pi/2} \int_a^{a(1+\cos\theta)} r\,dr.\,d\theta = 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} [a^2(1+\cos\theta)^2 - a^2]\,d\theta$$

$$= \int_0^{\pi/2} a^2 (1 + 2\cos\theta + \cos^2\theta - 1)\, d\theta$$

$$= a^2 \left[2 \int_0^{\pi/2} \cos\theta . d\theta + \int_0^{\pi/2} \cos^2\theta\, d\theta \right.$$

$$= a^2 \left[2(1) + \frac{1}{2}\left(\frac{\pi}{2}\right)\right]$$

$$I = \frac{a^2}{4}(\pi + 8) \text{ sq. units.}$$

Triple Integration

Problems on Triple Integration:

1) Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$.

Soln:

The correct form of the integral is

$$I = \int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx = \int_0^{\log 2} \int_0^{x} \left[e^{x+y+z} \right]_0^{x+y} dy\, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y+x+y} - e^{x+y} \right] dy\, dx = \int_0^{\log 2} \int_0^{x} \left[e^{2x+2y} - e^{x+y} \right] dy\, dx$$

$$= \int_0^{\log 2} \left[\frac{e^{2x+2y}}{2} - e^{x+y} \right]_0^{x} dx$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x} \right]_0^{\log 2} = \left[\frac{e^{4x}}{8} - \frac{3}{4} e^{2x} + e^{x} \right]_0^{\log 2}$$

$$= \frac{e^{4\log 2}}{8} - \frac{3}{4} e^{2\log 2} + e^{\log 2} - \frac{1}{8} + \frac{3}{4} - 1$$

$$= \frac{e^{\log 2^4}}{8} - \frac{3}{4} e^{\log 2^2} + e^{\log e} - \frac{1}{8} + \frac{3}{4} - 1$$

$$= \frac{e^{\log 2^4}}{8} \cancel{-\frac{3}{}} = \frac{2^4}{8} - \frac{3}{4}(2^2) + 1 - \frac{1}{8} + \frac{3}{4}$$

$$I = \frac{5}{8}.$$

2) Evaluate $\int xyz\, dz\, dy\, dx$ over the volume enclosed by the three co-ordinate planes and the plane $x+y+z=1$. [or]

Evaluate $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz\, dx\, dy\, dz$.

Soln;

we have to evaluate the given integral bounded by the planes ⟵ (1).

$x=0, y=0, z=0$ & $x+y+z=1$.

(1) $\Rightarrow z = 1-x-y$.

$\therefore$ z varies from $z=0$ to $z=1-x-y$.

Put $z=0$ in (1).

$x+y=1$ —— (2)

$\Rightarrow y = 1-x$.

$\therefore$ y varies from $y=0$ to $y=1-x$.

put $y=0$ in (2) $\Rightarrow x=1$

$\therefore$ x varies from $x=0$ to $x=1$.

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} xy\, dy\, dx.$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2\, xy\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{t} (t-y)^2\, xy\, dy\, dx.$$

$$= \frac{1}{2} \int_0^1 \int_0^{t} (t^2 - 2ty + y^2)\, xy\, dy\, dx$$

Put $1-x=t$

Don't find dt, because 't' is a constant w.r.to y.

$$= \frac{1}{2}\int_0^1\int_0^t (t^2y - 2ty^2 + y^3)\, x\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{t^2y^2}{2} - \frac{2ty^3}{3} + \frac{y^4}{4}\right]_0^t x\, dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{t^4}{2} - \frac{2t^4}{3} + \frac{t^4}{4}\right] x\, dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{12t^4 - 16t^4 + 6t^4}{24}\right] x\, dx = \frac{1}{24}\int_0^1 t^4 x\, dx$$

$$= \frac{1}{24}\int_0^1 x(1-x)^4\, dx = \frac{1}{24}\left[x\left[\frac{(1-x)^5}{-5}\right] - (1)\frac{(1-x)^6}{30}\right]_0^1$$

$$= \frac{1}{24}\left[0 - 0 - 0 + \frac{1}{30}\right]$$

$$I = \frac{1}{720}.$$

Volume as Triple Integral.

Problems on volume as Triple Integral:

1) Find the volume of the region bounded by $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Soln: ~~To plo~~

Step:1) To plot the region:

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is plane which intersed the coordinate axes at $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Step:2 To evaluate the integral:

$$V = \iiint dz\, dy\, dx.$$

Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ——— ① $\Rightarrow z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$\therefore$ z varies from $z=0$ to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$.

Put $z=0$ in (1) $\Rightarrow \frac{x}{a}+\frac{y}{b}=1$ ——— (2)

$\Rightarrow y = b\left(1 - \frac{x}{a}\right)$

$\therefore$ y varies from $y = 0$ to $y = b(1 - x/a)$

Put $y = 0$ in (2) $\Rightarrow \frac{x}{a} = 1 \Rightarrow x = a$.

$\therefore$ x varies from $x = 0$ to $x = a$.

Volume $V = \iiint dz\,dy\,dx$

$$V = \int_{0}^{x=a} \int_{0}^{y=b(1-\frac{x}{a})} \int_{z=0}^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-x/a)} [z]_0^{c(1-x/a-y/b)} dy\,dx$$

$$= \int_0^a \int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx$$

Put $t = 1 - x/a$; dont find dt because $1 - x/a$ is a constant w.r.to y.

$$\text{Volume } V = c\int_0^a \int_0^{bt} (t - y/b)\,dy\,dx = c\int_0^a \left(ty - \frac{y^2}{2b}\right)_0^{bt} dx.$$

$$= c\int_0^a \left(bt^2 - \frac{b^2t^2}{2b}\right)dx = c\int_0^a \frac{bt^2}{2}dx$$

$$= \frac{bc}{2}\int_0^a \left[1 - \frac{x}{a}\right]^2 dx = \frac{bc}{2}\int_0^a \left(1 - 2\frac{x}{a} + \frac{x^2}{a^2}\right)dx$$

$$= \frac{bc}{2}\left[x - 2\frac{x^2}{2a} + \frac{x^3}{3a^2}\right]_0^a = \frac{bc}{2}\left(a - a + \frac{a^3}{3a^2}\right)$$

Volume $= \frac{abc}{6}$ cubic units.

X ———— X